Data-enabled learning, network effects and competitive advantage*

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Abstract

We provide a model of competition when firms can improve their products through learning from the data they obtain on customers they serve. The model is used to explore the implications for competitive dynamics of three new features of such learning compared to traditional learning-by-doing settings: (i) learning increases a firm's demand rather than reducing its marginal cost, (ii) firms can improve their products for individual customers based on each customer's particular usage experience, and (iii) the learning happens while a firm's customers are still consuming the product. We show when and how network effects arise from these new features. We also analyze the role of consumer beliefs, the nature of the learning curve, and other factors that affect an incumbent's competitive advantage.

Keywords: big data, machine learning, firm competition, switching costs

1 Introduction

In recent years, much attention has been focused on the role data can play in providing incumbent firms with a competitive advantage. Digitization, connectivity to cloud-based infrastructures, together with cheaper storage and more effective use of data (i.e. improvements in machine learning algorithms), have made it possible for firms in many industries to learn from their customer data and rapidly improve their products. And with better products, these firms can attract more customers and therefore obtain more data, potentially creating a self-reinforcing cycle that can make it difficult for any new entrant to compete. Google Maps provides a clear example. As its traffic predictions improved, more drivers

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adopted it, resulting in more data to improve traffic predictions, more drivers adopting, and so on. We call this self-reinforcing cycle "data-enabled learning".[1] Other notable examples that exhibit data-enabled learning include search engines (Google, Bing, Baidu), speech recognition applications (e.g. dictation software, virtual assistants, chatbots), smart connected devices (e.g. the Nest thermostat, smart beds by Eight Sleep, the Tonal home gym), online recommender systems (e.g. Netflix, Stitchfix, Spotify, Tinder etc.), advanced driver assistance and autonomous vehicle systems (Cruise, Mobileye, Waymo), as well as a myriad of companies developing solutions for AI-based health monitoring and diagnostic procedures (e.g. Notable Labs and SkinVision), and AI-based agricultural robotic farming solutions and crop management (e.g. Blue River Technology and Prospera).

Data-enabled learning would seem to give incumbent firms a strong competitive advantage. But how is the competitive advantage obtained from data-enabled learning different from that obtained from more traditional mechanisms, such as (i) learning-by-doing, in which an incumbent lowers its production costs as it produces (and sells) more, thereby being able to offer a lower price, and attract more sales, and (ii) standard network effects, in which an incumbent firm becomes more valuable to consumers as it attracts more consumers, thereby attracting even more consumers and offering even higher value?

To address these questions, and determine the drivers of competitive advantage with data-enabled learning, we develop a theory of dynamic competition in which firms can improve their products via data gleaned from their customers. The theory is based on competition between two firms, which we can think of as an incumbent firm that has had some prior learning and an entrant that has not (or has less such prior learning). We allow firms to set prices to compete for consumers every period, taking into account how attracting consumers (and therefore more data) increases the value they can offer to consumers in subsequent periods. Firms are assumed to be identical other than any difference in the standalone utility they offer and any difference in value attributed to differences from learning. We allow for very general learning technologies — the only assumptions imposed on the firms' learning curves is that they are (weakly) increasing and bounded by some power function in the limit. Consumers are fully rational, forward looking, and have no exogenous switching costs. An attractive feature of our setup is that, despite the generality of our learning functions, we are able to obtain closed form solutions for the equilibrium outcomes, determining which firm will win and each firm's value for any arbitrary starting position of the two firms.

As a baseline, we consider the case with across-user learning only, in which the two firms

[1] Some practitioners have called this phenomenon "data network effects". Our use of "data-enabled learning" is intended to be more general since, as discussed below, the phenomenon need not involve any network effect (for example, if the learning is specific to each customer). Indeed, one of the contributions of this paper is to determine when data-enabled learning creates genuine network effects.

only learn based on the total number of consumers they served in the past. This differs from traditional learning-by-doing because learning works through willingness-to-pay rather than costs, meaning the dynamics of prices are quite different. Moreover, consumers' willingness-to-pay can increase over time with more users even if prices remain unchanged, so there is a type of dynamic network effect at play which is not present under learning-by-doing since in the case of learning-by-doing, consumers are only better off via lower prices (i.e. a pecuniary externality).

We then consider the case of within-user learning only, in which any given consumer's willingness-to-pay for a given firm's product only depends on how many times that firm has served that particular consumer in the past. This highlights a second key difference between data-enabled learning and traditional mechanisms, which is the role of customization: firms can often improve their products for each individual customer based on that customer's particular usage experience. Within-user learning implies the more a customer uses a firm's product, the greater is the effective cost of switching to use the rival's product. Such data-enabled learning therefore creates an endogenous switching costs that grows over time, but not any network effect.

In these baseline models, we show that despite the fact that sometimes both the winning firm and the losing firm price below cost to consumers, the competitive outcome always coincides with the socially optimal outcome. This is not at all an obvious result despite our Bertrand competition setting: the determination of which firm wins in a particular period depends on how much each firm is willing to subsidize consumers in the current period, which in turn depends on how much each firm's value increases by winning this period as opposed to losing this period, both of which are only defined recursively. A second fundamental result comes from the comparison of the case with pure across-user learning and the one with pure within-user learning. When the learning functions and standalone utilities are the same in the two models, the conditions for either firm to win are identical in the two cases. However, the winning firm always makes higher profits in the case with pure across-user learning relative to the case with pure within-user learning. We also explain how various factors, such as how far each firm is along its learning curve and the nature of the learning curve, affect the condition for either firm to win.

In these two baseline models, consumers do not face any coordination problem arising from network effects. Our third main contribution consists in showing how a coordination problem for consumers arises from network effects when we take into account a third distinguishing feature of data-enabled learning, which is the speed of learning and product improvement. Specifically, with digital cloud-based products and services, firms can often improve their products while their customers are still consuming them. When this happens within each period and is combined with across-user learning, consumers need to form expectations over how many other consumers will purchase the same product in order to determine the value they will get from the product in the current period. In this case, the incumbent's competitive advantage depends on consumer beliefs. We show the same is true in a setting where consumers derive utility over multiple periods from a given product purchase even if the learning only happens with a delay. Then the price they pay for their chosen product acts as an endogenous switching cost, which means that once again they have an incentive to choose the product that other consumers are expected to choose in subsequent periods.

Finally, we show that endogenous network effects in which consumers face a coordination problem arise even in the absence of any within-period learning, exogenous switching costs or prices, if we combine both across-user and within-user learning. Specifically, even if the product is available for free and consumers are free to change firms each period, within-user learning creates an endogenous switching cost. A consumer's choice of product in the current period affects the value they get from each firm in subsequent periods due to within-user learning, which in turn means they also want to choose the product that other consumers are expected to choose in order to also benefit from across-user learning in subsequent periods.

A key implication of our analysis is that even though data can create a self-reinforcing cycle for an incumbent firm, as we noted above, unless consumers face a coordination problem due to network effects, the outcome is efficient. Intervening in a way that pushes consumers to use the entrant's product would not be socially optimal taking into account the welfare losses that would be incurred along the path to this alternative outcome. Only when true network effects are present and consumer beliefs are biased in favor of one firm (e.g. the incumbent), does the equilibrium outcome deviate from the socially efficient outcome. We also use the model to show how the shape of the learning curve and each firm's position on it determine the extent of this distortion, as well as discuss the implications of data-enabled learning for consumer outcomes.

The rest of the paper proceeds as follows. In Section 2 we discuss related literature. In Section 3 we analyze a baseline model with across-user learning but no within-user learning and no within-period learning. In Section 4 we explore what happens when instead of across-user learning there is within-user learning. Beliefs and network effects do not play any role in either of these two models. In Section 5, we show how network effects in which beliefs do matter can endogenously arise once we introduce (i) within-period learning in a model with across-user learning, (ii) endogenous switching costs via a price paid for a durable product when there is across-user learning, (iii) both within-user and across-user learning. In each of these last three cases we contrast the equilibrium outcome under Pareto beliefs with the equilibrium outcome under beliefs favoring the incumbent firm.

2 Related literature

Our paper combines aspects from three large and established literatures. First, it relates to works that provide theoretical models of competition with learning-by-doing (Fudenberg and Tirole, 1983, Dasgupta and Stiglitz, 1988, Cabral and Riordan, 1994, Besanko et al., 2010). Learning-by-doing models typically assume firms' marginal costs are decreasing in past sales. Aside from the differences between data-enabled learning and learning-by-doing highlighted in the introduction, our particular model of learning is quite distinct from any in the existing literature. The baseline model with pure across-user learning is closest to the classic paper of Cabral and Riordan (1994). Key differences are that firms in our setting are asymmetric from the start and consumers do not view them as horizontally differentiated. An important benefit of our particular framework is tractability: it allows us to obtain explicit expressions for the price paths of each firm, the conditions determining which firm wins, and the firms' values, as well as corresponding comparative statics. This allows us to apply the framework to the case of within-user learning and to accommodate within-period learning—both are novelties in this literature. Finally, the efficiency result of our equilibrium outcome should be of interest to those interested in learning-by-doing, since the same result would also hold in an equivalent learning-by-doing setting.

Second, our paper relates to the substantial literature on network effects. From this literature we borrow the modelling of consumer "beliefs" to select equilibria when consumers play coordination games. This literature started with the pioneering work of Katz and Shapiro (1985), and different beliefs have been explored by Katz and Shapiro (1986), who apply the notion of "Pareto beliefs" (given consumers are identical, consumers coordinate on the equilibrium which offers the highest surplus for them), and Caillaud and Jullien (2001, 2003), who apply the notion of "favorable beliefs" (consumers coordinate on the equilibrium which favors one particular firm).

Third, our paper relates to IO models of switching costs, such as the classic papers of Klemperer (1987a,b) and Beggs and Klemperer (1992). As noted in the introduction, within-user learning creates an endogenous switching cost that grows over time. To the best of our knowledge, such a feature has not previously been considered in the switching cost literature.

Finally, there are some recent works that also study data-enabled learning, albeit they take a very different approach from ours. Prufer and Schottmüller (2017) model a dynamic feedback loop in which a firm's current cost of investing in quality is decreasing in the firm's previous period sales, and greater investment in quality leads to higher demand in the current period. This differs from our approach in a few ways. They do not explicitly model consumers' choices or firms' pricing decisions, but focus instead on firms' investment decisions. While they allow for learning across users, they do not consider either within-period learning or within-user learning, and so user coordination issues do not arise. Finally, they explore the interesting issue of how learning from data allows the dominant firm to enter connected markets when the existing data the firm has collected can be reused, an issue we do not consider. Somewhat similarly, Farboodi et al. (2019) provide a model of data-enabled learning in which data helps firms choose the best production technique and so higher quality products. They abstract from any strategic firm behavior or consumer decisions, and instead focus on the industry dynamics generated from a continuum of price-taking firms. De Cornière and Taylor (2020) also study the effect of data on competition but focus on showing how different uses of data (including improving a firm's product, but also ad targeting and price discrimination) determine whether data is pro- or anti-competitive. While their framework is much more general, they take a more reduced-form approach (e.g. they don't work with the underlying learning functions), and furthermore, their main setup is static, whereas ours is fully dynamic. Schaefer et al. (2018) investigate how data drives the quality of internet search results, providing evidence that the quality of search results improve with more data on previous searches, and that personalized information is particularly valuable. Finally, Biglaiser et al. (2019) provides a high-level discussion of the different ways firms can enjoy incumbency advantages, including access to more data: they note that across-user learning and within-user learning are the two distinct ways in which data generates a competitive advantage for incumbents.

3 Across-user learning

In this section we consider pure across-user learning. This means each firm improves the product for each consumer based on what it learns from the usage of all its consumers. Put differently, the learning is not specific to any individual user. For example, traffic information in Google Maps updates based on information collected from all drivers, and the traffic information shown at any point in time is likely to be the same (or very similar) for everyone using Google Maps. This contrasts with within-user learning (which we will look at in the next section), where the learning allows the product to be improved for a specific user, the more they use it (e.g. speaker-dependent voice recognition).

The learning technology we consider has the general property that the value to consumers of a given firm's product in a period is an increasing function of the measure of consumers who have purchased the product previously (past consumers) from the same firm. In this section, we assume firms can only incorporate learning from current customers in their products starting next period. This shuts down any role for beliefs and network effects in the equilibrium we analyze. We will explore the implications of allowing for "within-period" learning in Section 5.1. As we will explain in that section, all of our results in this section can be easily adjusted to allow for within-period learning provided we focus on certain types of consumer beliefs, where the specific adjustment depends on the nature of these beliefs.

Two firms, which we refer to as the incumbent I and the entrant E, compete over infinitely many periods. Both firms face a marginal cost of production equal to c and compete in prices. We assume consumers have unit demand in each period and c is high enough that prices are never negative, thus ensuring consumers would only ever want to purchase one product in a given period. The (common) one period ahead discount factor is denoted δ, and satisfies $0 < \delta < 1$. Like Cabral and Riordan (1994), we normalize the measure of consumers in each period to one, and so the number of periods that a firm has previously won also represents the total measure of consumers it can learn from. In this formulation of across-user learning, there is no distinction between consumers who are purchasing again (repeat consumers) vs. entirely new consumers. Thus, our setting here is compatible with two interpretations:

- Consumers live for one period only. New consumers in a period make a single buying decision and then exit.

- The same consumers are active and make purchasing decisions in every period, i.e. they are infinitely lived, but there is no cost for each of these individual consumers to switch from one firm to the other across periods. In this case, the learning from consumers when they purchase again is assumed to be the same as if they were new consumers.

We assume all consumers are atomistic, which is reasonable for most of our examples with across-user learning. Given our assumptions on consumers, this means even if consumers are of the second type above, they do not need to consider the impact of their purchase decision on the pricing game or the options they will face in subsequent periods. Later, when we look at within-user learning (or more generally, endogenous switching costs), this will no longer be true.

The standalone value of firm i's product (or service) is s_i, where $i \in \{I, E\}$. This value is augmented by learning. In any period, given our asymmetric Bertrand competition framework, each firm will either sell to all (measure one) of the consumers or none. As a result, the measure of consumers a firm has learnt from is equivalent to the number of periods in which it has made positive sales. If firm i has sold for N_i periods in the past, then the value current consumers obtain from firm i's product is $s_i + f_i\left(\min\left\{N_i, \overline{N}_i\right\}\right)$, where $\overline{N}_i$ is the number of periods firm i must sell for in order to reach its maximum threshold of learning.[2]

The idea is firms learn more as they obtain more consumer data, and this allows them to increase the value of the product they offer, but at some point they cannot use consumer data to increase the value of the product offered any further (i.e. learning is bounded). Later in this section we show that our model and results can also handle unbounded learning functions as limit cases provided we impose the very weak requirement that they can be bounded by some power function in the limit. The only requirements on firm i's learning function f_i are that it is everywhere weakly increasing, and without generality, strictly increasing in the last step to reach its maximum threshold (i.e. $f_i(\overline{N}_i) > f_i(\overline{N}_i - 1)$). In particular, f_i need not be continuous or take a particular shape (e.g. concave or convex). Thus, at any point in time there are potentially three sources of asymmetry between firms: (i) differences in standalone value ($s_I \neq s_E$), (ii) differences in the learning curve ($f_I \neq f_E$ and/or $\overline{N}_I \neq \overline{N}_E$), and (iii) differences in how far along each firm is on its learning curve in the current period ($N_I \neq N_E$).

Consumers and firms have perfect and complete information. Our focus is on Markovian strategies and a Markov Perfect Equilibrium (MPE), i.e. a subgame perfect equilibrium such that each player's strategy only depends on the state of the game, thus ruling out tacit collusion and other reputation-based strategies. Among MPEs, we also rule out equilibria which are supported by the losing firm pricing in such a way that it would make a loss if it instead won consumers at the prices charged in a particular period and then firms followed their equilibrium pricing strategies in all subsequent periods. This is equivalent to the standard approach of focusing on a Nash equilibrium in which the losing firm does not price below its marginal cost in the one-shot asymmetric Bertrand game. Thus, when we refer to a unique MPE we mean the MPE is unique after eliminating MPEs that involve such weakly dominated strategies.

The relevant state at the start of any period is then defined only by the measure of consumers each firm has previously served, i.e. by (N_I, N_E). The corresponding value functions for I and E (i.e. the present discounted values of future profit flows starting with the current period when firms follow their equilibrium pricing strategies in every period) are denoted by $V^I(N_I, N_E)$ and $V^E(N_I, N_E)$.

Consider firms starting from the state $(\overline{N}_I, \overline{N}_E)$, so both firms have reached their respective learning thresholds. Then I offers $s_I + f_I(\overline{N}_I)$ to consumers in every period and E offers $s_E + f_E(\overline{N}_E)$ to consumers in every period. Given the resulting asymmetric Bertrand competition, and taking into account that the competitive situation will be the same in

[2] In what follows, since it will be equivalent, we will refer to the number of periods of positive sales as the measure of consumers the firm has sold to.

every subsequent period, E wins every period starting with the current one if and only if $s_E + f_E(\overline{N}_E) \geq s_I + f_I(\overline{N}_I)$. Furthermore, for each firm $i \in \{I, E\}$, we must have

$$V^i(\overline{N}_I, \overline{N}_E) = \frac{1}{1-\delta} \max \left\{ s_i - s_j + f_i(\overline{N}_i) - f_j(\overline{N}_j), 0 \right\},$$

where $i \neq j$.

When a firm has not yet reached its learning threshold, in order to win in the current period it will be willing to offer a subsidy equal to the difference in the discounted value of its future profits if it were to win the current period and its future profits if it were to lose the current period. Taking into account the maximum subsidy each firm is willing to offer in order to win, we can work out which firm will win in each period, working backwards from $(N_I, N_E) = (\overline{N}_I, \overline{N}_E)$. Using this approach, we show that for any state (N_I, N_E), there is a threshold level of $s_E - s_I$ above which E wins and below which I wins. Moreover, along the equilibrium path, if a firm wins in the current period, because it benefits from an additional learning period, the threshold shifts such that the same firm will continue to win in each subsequent period. We will use the convention that E wins in case consumers are indifferent when both firms offer their maximum subsidy in order to win. Then for all $0 \leq N_I \leq \overline{N}_I$ and $0 \leq N_E \leq \overline{N}_E$, define

$$\begin{aligned} \Delta(N_I, N_E) \;&=\; (1-\delta)\left(\sum_{j=0}^{\overline{N}_I - N_I} \delta^j f_I(N_I + j) - \sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E + j)\right) \\ &\quad + \delta^{\overline{N}_I - N_I + 1} f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E + 1} f_E(\overline{N}_E). \end{aligned} \tag{1}$$

This function is proportional to the difference in gross surplus generated from learning across the two firms, comparing the paths where each firm wins in every period from the current period onwards. Note that $\Delta(N_I, N_E)$ is increasing in N_I since

$$\Delta(N_I + 1, N_E) - \Delta(N_I, N_E) = (1-\delta) \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j (f_I(N_I + 1 + j) - f_I(N_I + j)) > 0,$$

where we have used that $f_I(\overline{N}_I) > f_I(\overline{N}_I - 1)$. And by symmetry, $\Delta(N_I, N_E)$ is decreasing in N_E.

In the appendix we use proof by induction to establish the following Proposition.

Proposition 1. *Suppose I has previously sold to $0 \leq N_I \leq \overline{N}_I$ consumers and E has previously sold to $0 \leq N_E \leq \overline{N}_E$ consumers. There exists a unique MPE in which E will win in all periods iff $s_E - s_I \geq \Delta(N_I, N_E)$, and I will win in all periods otherwise, where*

$\Delta(N_I, N_E)$ is defined by (1). The equilibrium outcome is socially optimal.

Proposition 1 characterizes I's competitive advantage, which is captured by $\Delta(N_I, N_E)$. The higher is $\Delta(N_I, N_E)$, the higher is the level of $s_E - s_I$ required for E to win. The present discounted value (PDV) of the cutoff (i.e. $\Delta(N_I, N_E)$ divided by $1-\delta$) is equal to (i) the difference between the PDVs of gross surplus from learning along I's path to maximum learning assuming it wins every period vs. learning along E's path to maximum learning assuming it wins every period (the expression in the first line in (1)), and (ii) the difference between the PDV of learning offered by I vs. E once maximal learning is achieved by both firms (the expression in the second line in (1)).

Before providing more interpretation of Proposition 1, note that the proposition was worked out assuming that there are some maximum learning thresholds $\overline{N}_I$ and $\overline{N}_E$ that allow us to pin down the equilibrium (i.e. that the learning curve is bounded by some fixed maximum level of learning). The next Proposition extends these results to the case of any increasing learning curves by taking the limit of our results as $\overline{N}_I \to \infty$ and $\overline{N}_E \to \infty$, showing that our framework can also allow for unbounded learning functions. A sufficient condition for this limit approach to be well defined is that the learning functions lie below some power function in the limit. This is a very weak requirement that allows us to capture any reasonable learning function. Thus, for example, our results continue to hold in the limit with everywhere-increasing linear learning curves or convex learning curves such as $f_i(x) = \theta_i x^{\rho_i}$, with $\rho_i > 1$ for $i = I, E$. Moreover, by the same logic underlying this result, all our subsequent characterizations of $\Delta(N_I, N_E)$, $V^I(N_I, N_E)$ and $V^E(N_I, N_E)$ also extend to this unbounded case, by taking the limit as $\overline{N}_I \to \infty$ and $\overline{N}_E \to \infty$.

Proposition 2. *Suppose I has previously sold to $0 \leq N_I < \overline{N}_I$ consumers, E has previously sold to $0 \leq N_E < \overline{N}_E$ consumers. The results of Proposition 1 apply in the limit as $\overline{N}_I \to \infty$ and $\overline{N}_E \to \infty$ (i.e. such that the learning functions continue to increase forever) provided there exists some power function which the learning curves lie below in the limit.*

Since (1) divided by $1-\delta$ corresponds to the difference in the PDV of expected learning value generated if I wins in every period vs. E wins in every period, the cutoff for E to win also turns out to be exactly the socially optimal rule for E to win, as stated in Proposition 1. This is not an obvious result despite the Bertrand competition setting. The determination of which firm wins in a particular period depends on the consumers' willingness-to-pay for each firm in the current period and on how much each firm can increase its value next period by winning as opposed to losing the current period. This increase in value is only recursively defined and represents how much each firm is willing to subsidize current consumers in order to win. This maximum subsidy simplifies when firms are sufficiently close to the cutoff point since then if either firm loses, its value next period will be zero (even off the equilibrium path). Thus, the determination of whether I will win comes down to whether the difference in willingness-to-pay across the two firms in the current period (i.e. $s_I - s_E + f_I(N_I) - f_E(N_E)$) and the discounted difference in the firms' values next period if each firm were to win (i.e. $\delta(V^I(N_I+1, N_E) - V^E(N_I, N_E+1))$) is positive. However, neither $V^I(N_I+1, N_E)$ nor $V^E(N_I, N_E+1)$ correspond to the future value created for consumers by the winning firm along its winning path because the winner has to compete with a loser that is willing to subsidize consumers for some number of periods in an attempt to win, which means the winning firm only captures some part of the future value created. However, it turns out $\delta(V^I(N_I+1, N_E) - V^E(N_I, N_E+1))$ gives an expression that, after summing together with the difference in willingness-to-pay across the two firms in the current period, is proportional to $s_I - s_E + \Delta(N_I, N_E)$. This is positive iff $s_E - s_I < \Delta(N_I, N_E)$, which corresponds to the socially optimal condition for I to win.

Further emphasizing the non-obvious nature of this finding, the cutoff for E to win in a finite period version of the model is not generally socially optimal. Indeed, in the Online Appendix we show that the equilibrium cutoff when the firms start with (N_I, N_E) previous sales and the game ends after $T \geq 1$ periods is

$$\Delta(N_I, N_E, T) = (1-\delta)\left(\sum_{j=0}^{T-1} \frac{\delta^j(1-\delta^{T-j})}{1-(T+1)\delta^T + T\delta^{T+1}}(f_I(N_I+j) - f_E(N_E+j))\right),$$

while the socially efficient cutoff is

$$\Delta^S(N_I, N_E, T) = (1-\delta)\left(\sum_{j=0}^{T-1} \frac{\delta^j}{1-\delta^T}(f_I(N_I+j) - f_E(N_E+j))\right).$$

Taking the difference,

$$\begin{aligned} & \Delta(N_I, N_E, T) - \Delta^S(N_I, N_E, T) \qquad (2) \\ = \quad & (1-\delta)\sum_{j=0}^{T-1}\delta^j\left(\frac{1-\delta^{T-j}}{1-(T+1)\delta^T + T\delta^{T+1}} - \frac{1}{1-\delta^T}\right)(f_I(N_I+j) - f_E(N_E+j)). \end{aligned}$$

Note that for $T = 1$, there is no distortion, i.e. $\Delta(N_I, N_E, 1) = \Delta^S(N_I, N_E, 1)$. However, when $T > 1$, we generally have $\Delta(N_I, N_E, T) \neq \Delta^S(N_I, N_E, T)$.

The term in large brackets in (2) is positive when $j = 0$, everywhere decreasing in j, and negative when $j = T-1$. Given that $\sum_{j=0}^{T-1}\delta^j\left(\frac{1-\delta^{T-j}}{1-(T+1)\delta^T+T\delta^{T+1}} - \frac{1}{1-\delta^T}\right) = 0$, the distortion depends on some weighted average of the difference in the learning functions at consecutive steps, where the weights sum to zero. Thus, in general for a finite number of periods greater than one, the distortion can go in either direction depending on the shapes of the learning curves. For example, with $f_I = f_E \equiv f$ and $N_I > N_E$, the distortion is positive if f is concave, negative if f is convex, and zero if f is linear. On the other hand, if $f_i(x) = \theta_i x$ and $N_I \geq N_E$, the distortion is positive if $\theta_I < \theta_E$, negative if $\theta_I > \theta_E$, and zero if $\theta_I = \theta_E$. This last result is somewhat surprising: it implies that in this finite case, when E learns faster than I, in equilibrium I wins too often relative to the efficient criterion. Finally, note that for any given j, the term in large brackets in (2) tends to zero as T tends to infinity,[3] which is consistent with the result in Proposition 1 that there is no distortion in the infinite horizon case.

To gain some intuition for why there is a distortion with a finite number of periods, note that in the last period, the winning firm only captures the additional surplus it creates relative to the losing firm, so the winning firm places less value on the last period than the full social surplus it creates. Given each period's pricing is driven by the future value a firm can obtain by winning vs. losing, this distortion feeds through to all earlier periods, which results in firms taking into account more surplus than socially efficient in earlier periods and less surplus than socially efficient in later periods. As the number of periods increases, the effect of this last-period distortion on earlier periods diminishes, and eventually vanishes in the limit.

As part of the proof of Proposition 1 we characterize the value function for each firm, which we summarize here.

Corollary 1. *Suppose I has previously sold to* $0 \leq N_I \leq \overline{N}_I$ *consumers and E has previously sold to* $0 \leq N_E \leq \overline{N}_E$ *consumers. In the unique MPE, the firms' value functions are determined as follows:*

- *if* $s_E - s_I < \Delta(N_I, N_E + 1)$ *then* $V^E(N_I, N_E) = 0$ *and* $V^I(N_I, N_E) = \frac{s_I - s_E}{1-\delta} + \frac{\Delta(N_I,N_E) - \delta\Delta(N_I,N_E+1)}{(1-\delta)^2}$ *(no subsidization by E)*
- *if* $\Delta(N_I, N_E + 1) \leq s_E - s_I < \Delta(N_I, N_E)$*, then* $V^E(N_I, N_E) = 0$ *and* $V^I(N_I, N_E) = \frac{s_I - s_E + \Delta(N_I,N_E)}{(1-\delta)^2}$ *(subsidization by E)*
- *if* $\Delta(N_I, N_E) \leq s_E - s_I < \Delta(N_I + 1, N_E)$*, then* $V^E(N_I, N_E) = \frac{s_E - s_I - \Delta(N_I,N_E)}{(1-\delta)^2}$ *and* $V^I(N_I, N_E) = 0$ *(subsidization by I)*

[3] It is also easily verified that when $f_i(N_i) = f_i\left(\min\left\{N_i, \overline{N}_i\right\}\right)$ for $i = I, E$ and T tends to infinity, both $\Delta(N_I, N_E, T)$ and $\Delta^S(N_I, N_E, T)$ converge to $\Delta(N_I, N_E)$ defined in (1).

- *if* $s_E - s_I \geq \Delta(N_I+1, N_E)$, *then* $V^E(N_I, N_E) = \frac{s_E - s_I}{1-\delta} - \frac{\Delta(N_I,N_E) - \delta\Delta(N_I+1,N_E)}{(1-\delta)^2}$ *and* $V^I(N_I, N_E) = 0$ *(no subsidization by I),*

where we define $\Delta(\overline{N}_I + 1, N_E) \equiv \Delta(\overline{N}_I, N_E)$ *for any* $0 \leq N_E \leq \overline{N}_E$ *and* $\Delta(N_I, \overline{N}_E + 1) \equiv \Delta(N_I, \overline{N}_E)$ *for any* $0 \leq N_I \leq \overline{N}_I$.

Corollary 1 characterizes the PDV of the firms' equilibrium profits for different states (N_I, N_E) and different ranges of the underlying parameters. An interesting implication of the Corollary is that the losing firm is still willing to subsidize consumers for some range of parameters (specifically, when the winning firm's competitive advantage is not too large). This reflects that in the off-equilibrium situation that the losing firm actually wins the current period, it expects to have a positive value (via some future profits). This implies the winning firm's competitive advantage will be limited by the losing firm's willingness to subsidize consumers in such cases. The willingness of a firm to offer a subsidy reflects the "prize" that the firm can obtain from winning, which is not only that it obtains an improvement in its product (or moves closer to obtaining this improvement) but that the other firm does not. This underlying logic is similar to that in standard learning-by-doing settings (e.g. Cabral and Riordan, 1994) but the price dynamics are quite different given the winning firm can charge more with data-enabled learning, whereas the winning firm charges less when learning reduces marginal cost under learning-by-doing settings.

If $s_E - s_I$ is large enough such that even if I wins it will not be able to profit in the future (i.e. $s_E - s_I \geq \Delta(N_I + 1, N_E)$ when the current state is (N_I, N_E)), or $s_E - s_I$ is low enough so that even if E wins it will not be able to profit in the future (i.e. $s_E - s_I < \Delta(N_I, N_E + 1)$ when the current state is (N_I, N_E)), the losing firm prices at c in all periods. These are the cases listed as "no subsidization" in the proposition, and in these cases the winning firm extracts the full additional value it offers relative to the losing firm (that prices at c) in each period. When $s_E - s_I$ is neither so high or so low that the losing firm gives up trying to win, the winning firm's value is reduced by the fact that it has to compete against a firm that is willing to subsidize to win. And the amount the winning firm has to give up depends on how close the losing firm is to the threshold.

Corollary 1 has important implications for how data-enabled learning impacts consumer surplus. Note that in equilibrium, the firms' prices in any given period leave consumers indifferent between the two firms. Since along the equilibrium path, once a firm wins, it will win in all future periods, this implies that the PDV of consumer surplus is determined by the surplus offered by the losing firm in each period along the equilibrium path. As a result, learning by the winning firm as it sells to additional consumers each period makes consumers weakly worse off. To see this note first that if the losing firm has already stopped offering a subsidy (e.g. the no-subsidization case in Corollary 1), then learning by the winner does not change the PDV of consumer surplus offered by the loser, which in case firm i is the losing firm is fixed at

$$CS\left(N_I, N_E\right) = \frac{s_i - c + f_i(N_i)}{1-\delta}.$$

In case the losing firm currently offers a subsidy (either one of the two subsidization cases in Corollary 1), the PDV of consumer surplus can be determined by the PDV of total surplus less the PDV of the winning firm's profit stream, i.e.

$$CS\left(N_I, N_E\right) = \frac{s_i - c}{1-\delta} + \sum_{k=0}^{\overline{N}_i - N_i} \delta^k f_i(N_i + k) + \frac{\delta^{\overline{N}_i - N_i + 1}}{1-\delta} f_i(\overline{N}_i) - \left(\frac{s_i - s_j + \Delta(N_I, N_E)}{(1-\delta)^2}\right)$$

in case firm i is the winning firm and firm j is the losing firm. It is straightforward to check that in this case $CS\left(N_I, N_E\right)$ is decreasing in N_i, so consumer surplus is strictly decreasing as the winning firm learns more along the equilibrium path. This reflects that the losing firm's subsidy gets smaller over time as the winning firm's learning advantage continues to increase along the equilibrium path.

The fact that additional learning by the winning firm as it sells to more consumers does not help consumers reflects the fact in this model, the winning firm extracts all the additional learning benefit above the surplus that the losing firm can offer. While with a more general model of competition or with elastic aggregate demand we may expect some of the winning firm's learning benefit to be shared with consumers, the result that consumers can be made worse off by the winning firm's learning because the losing firm is no longer willing to subsidize as much in an attempt to compete does seem more general. It implies consumers would be better off if they could somehow coordinate on joining the losing firm instead of the winning firm for a number of periods, at least until the losing firm would only require to win one more period of learning to become the winning firm. However, given each individual consumer is atomless and can freely switch in each period, it has no reason to internalize these effects since its own decision about which firm to join will have no effect on the consumer surplus it can obtain next period (in the case it is long lived). This suggests there is a possible role for policy, either to help coordinate consumers decisions or to help the losing firm catch up. The latter could involve requiring the winning firm to share some of its data (or learning) with the losing firm.

Corollary 1 also allows us to say how many more periods of sales a losing firm is willing to continue subsidizing consumers until it gives up subsidizing and just prices at cost, which we fully characterize in the next corollary. Remarkably, for some range of parameter values (when the losing firm's intrinsic product advantage is sufficiently strong), the losing firm will continue offering a subsidy forever.

Corollary 2. *Suppose I has previously sold to* $0 \le N_I \le \overline{N}_I$ *consumers and E has previously sold to* $0 \le N_E \le \overline{N}_E$ *consumers. Then the number of periods the losing firm subsidizes for is determined as follows:*

- *if* $s_E - s_I < \Delta(N_I, N_E + 1)$*, then I wins, E gives up right away and never subsidizes*
- *if* $\Delta(N_I, N_E + 1) \le s_E - s_I < \min\left\{\Delta\left(\overline{N}_I, N_E + 1\right), \Delta\left(N_I, N_E\right)\right\}$*, then I wins and E subsidizes for the first k periods, where* $k \le \overline{N}_I - N_I$ *is uniquely defined by the inequality* $\Delta(N_I + k, N_E + 1) < s_E - s_I < \Delta(N_I + k + 1, N_E + 1)$
- *if* $\Delta(\overline{N}_I, N_E + 1) \le s_E - s_I < \Delta(N_I, N_E)$*, then I wins and E subsidizes forever, i.e. even after I reaches the threshold*
- *if* $\Delta(N_I, N_E) \le s_E - s_I < \Delta(N_I + 1, \overline{N}_E)$*, then E wins and I subsidizes forever, i.e. even after E reaches the threshold*
- *if* $\max\left\{\Delta(N_I + 1, \overline{N}_E), \Delta(N_I, N_E)\right\} \le s_E - s_I < \Delta\left(N_I + 1, N_E\right)$*, then E wins and I subsidizes for the first k periods, where* $k \le \overline{N}_E - N_E$ *is uniquely defined by the inequality* $\Delta\left(N_I + 1, N_E + k + 1\right) < s_E - s_I < \Delta\left(N_I + 1, N_E + k\right)$
- *if* $s_E - s_I \ge \Delta(N_I + 1, N_E)$*, then E wins, I gives up right away and never subsidizes,*

where we have once define $\Delta\left(\overline{N}_I + 1, N_E\right) \equiv \Delta(\overline{N}_I, N_E)$ *for any* $0 \le N_E \le \overline{N}_E$ *and* $\Delta\left(N_I, \overline{N}_E + 1\right) \equiv \Delta(N_I, \overline{N}_E)$ *for any* $0 \le N_I \le \overline{N}_I$*.*

To understand the role of prices in Proposition 1, note that if prices were constrained to be zero in every period, there would be no subsidies. Then consumers would compare firms solely on the basis of the surplus each offers in the current period. This means consumers would compare $s_I + f_I(N_I)$ and $s_E + f_E(N_E)$, and so choose E if and only if $s_E - s_I \ge f_I(N_I) - f_E(N_E)$. Clearly consumers do not take into account the future learning benefits in this case, which is why without prices, the equilibrium choice of firm would diverge from the socially optimal one characterized in Proposition 1. This emphasizes that competitive subsidies play a key role in ensuring atomistic consumers who make their purchase decisions in every period solely on the basis of current period utility (because even in the case that they are long-lived, they face no cost of switching firms in any future period and can ignore the implications of their individual choice on the future options they will face), end up taking into account the future benefits of their choices in deciding which firm to buy from.

3.1 Special cases

To further interpret Proposition 1, we characterize the cutoff for E to win for a number of special cases. We will use some of these to obtain specific comparative static results in Section 3.2.

Example 1: Identical learning functions. Suppose both firms face the same learning function. Formally, this means $f(x) \equiv f_I(x) = f_E(x)$ for any x, and $\overline{N} \equiv \overline{N}_I = \overline{N}_E$. Then (1) implies

$$\begin{aligned}\Delta(N_I, N_E) &= (1-\delta)\left(\sum_{j=0}^{\overline{N}-N_I} \delta^j f(N_I+j) - \sum_{j=0}^{\overline{N}-N_E} \delta^j f(N_E+j)\right) \\ &\quad + \left(\delta^{\overline{N}-N_I+1} - \delta^{\overline{N}-N_E+1}\right) f\left(\overline{N}\right).\end{aligned} \tag{3}$$

It is straightforward to confirm that $\Delta(N_I, N_E) > 0$ if and only if $N_I > N_E$, so in this case, the firm further along the learning curve enjoys a competitive advantage.

Example 2. Linear learning function. Suppose $f_i(x) = \theta_i x$, which implies if firm i has sold to N_i past consumers, the value it offers from learning is $f_i\left(\min\left\{N_i, \overline{N}_i\right\}\right) = \theta \min\left\{N_i, \overline{N}_i\right\}$. Then (1) implies

$$\Delta(N_I, N_E) = N_I\theta_I - N_E\theta_E + \frac{(\delta - \delta^{\overline{N}_I - N_I+1})\theta_I - (\delta - \delta^{\overline{N}_E - N_E+1})\theta_E}{1-\delta}. \tag{4}$$

This highlights that I's competitive advantage depends on how many periods each firm has already learnt for (i.e. N_I vs. N_E), how close each firm is to their respective maximum learning threshold ($\overline{N}_I - N_I$ vs. $\overline{N}_E - N_E$), and the value they create each period from additional learning (θ_I vs. θ_E). If $f_i(x) = \theta x$ and $\overline{N}_I = \overline{N}_E$ so the two learning curves are identical, then this simplifies to

$$\Delta(N_I, N_E) = \left(N_I - N_E - \frac{\delta^{\overline{N}-N_I+1} - \delta^{\overline{N}-N_E+1}}{1-\delta}\right)\theta. \tag{5}$$

Suppose $N_I > N_E$. I obtains an initial learning advantage of $(N_I - N_E)\theta$. This is adjusted downward by a partially offsetting effect reflecting that E can benefit consumers more by winning compared to I once I has already reached the maximum learning threshold but E has not, since by so doing it is able to close the learning gap. The extent of this benefit depends on how many additional periods it takes E to reach the threshold, and how far in the future this scenario arises.

If we take $N_E = 0$ and $N_I > 0$ to reflect I's initial advantage in (4), then taking the limit as $\overline{N}_I \to \infty$ and $\overline{N}_E \to \infty$ implies the cutoff determining whether E will win is simply $N_I\theta_I + \frac{\delta(\theta_I - \theta_E)}{1-\delta}$ in the asymmetric case. I's competitive advantage just depends on comparing its initial learning advantage from its additional N_I past periods of learning with any higher rate at which E learns relative to I. This limit case illustrates Proposition 2 for the case of an unbounded linear learning curve, with the cutoff and value functions still being well defined for any $0 < \delta < 1$.

Example 3: General S-shaped learning function. A more realistic learning curve takes an S-shape, with value initially increasing little with additional data since it takes some initial minimum corpus of data to get things started, a period of rapid value creation from additional data once the minimum training data has been gathered, and then value no longer increasing much with additional data as most learning possible from data has been exhausted. We can approximate such a learning curve by assuming there is no learning until a minimum amount of past sales is obtained, after which there is linear learning until the maximum threshold level of learning is achieved. Let the minimum threshold be denoted $\underline{N}$, where $0 \leq \underline{N} \leq \overline{N}$. This means that there is no learning until the firm has sold to more than $\underline{N}$ consumers in the past. Formally, the common learning function is $f(x) = \theta \max\{x - \underline{N}, 0\}$, which implies if firm i has sold to N_i past consumers, the value it offers from learning is $f_i\left(\min\left\{N_i, \overline{N}\right\}\right) = \theta \max\left\{\min\left\{N_i, \overline{N}\right\} - \underline{N}, 0\right\}$. Thus, the learning curve is flat up until the number of past customers exceeds $\underline{N}$, increasing, and flat again once the number of past customers exceeds $\overline{N}$.

Applying this function to (3) when $N_I < \underline{N}$ and $N_E < \underline{N}$, so neither firm has reached the minimum threshold $\underline{N}$ yet, the cutoff for E to win is

$$\Delta(N_I, N_E) \equiv \left(\frac{\delta^{\underline{N}-N_I+1} - \delta^{\overline{N}-N_I+1}}{1-\delta} - \frac{\delta^{\underline{N}-N_E+1} - \delta^{\overline{N}-N_E+1}}{1-\delta} \right) \theta. \tag{6}$$

This captures the difference in the value expected to be offered by the respective firms in the future once they move onto the increasing part of the learning curve, taking into account the discounting that applies given this additional value only arises in the future.

Combining the results of (5) and (6), so that we can allow neither, one or both firms to have reached the minimum threshold $\underline{N}$ already, the cutoff for E to win becomes

$$\begin{aligned}\Delta(N_I, N_E) &\equiv (E(N_E) - E(N_I))\theta \\ &+ \left(\frac{\delta^{\overline{N}-N_I+1-E(N_I)} - \delta^{\overline{N}-N_I+1}}{1-\delta} - \frac{\delta^{\overline{N}-N_E+1-E(N_E)} - \delta^{\overline{N}-N_E+1}}{1-\delta} \right)\theta,\end{aligned} \tag{7}$$

where $E(N_I) \equiv \overline{N} - \max\{N_I, \underline{N}\}$ and $E(N_E) \equiv \overline{N} - \max\{N_E, \underline{N}\}$ for all $0 \leq N_I \leq \overline{N}_I$ and $0 \leq N_E \leq \overline{N}_E$.

A special case of Example 3 arises when $\overline{N} - \underline{N} = 1$ and $\underline{N} > 0$, so there is no learning until the first threshold is reached, after which maximum learning (equal to θ) is immediately obtained. We refer to this as a *step* learning function. Then imposing these conditions on (6) implies the cutoff for E to win becomes

$$\Delta(N_I, N_E) \equiv (\delta^{\overline{N}-N_I} - \delta^{\overline{N}-N_E})\theta.$$

3.2 Determinants of competitive advantage

Recall I's competitive advantage is measured by $\Delta(N_I, N_E)$ since a higher value of $\Delta(N_I, N_E)$ means E requires a higher standalone benefit to win (or equivalently I can win with a lower standalone benefit). Some general comparative static results are summarized in the following Proposition.

Proposition 3. *Consider the general learning functions f_I and f_E. Suppose $0 \leq N_I \leq \overline{N}_I$ and $0 \leq N_E \leq \overline{N}_E$. Then I's competitive advantage is (i) increasing in the number of previous consumers N_I that I has sold to and decreasing in the number of previous consumers N_E that E has sold to; (ii) increasing in the maximum threshold for I's learning $\overline{N}_I$ and decreasing in the maximum threshold for E's learning $\overline{N}_E$; (iii) increasing in anything that increases I's learning function f_I and decreasing in anything that increases E's learning function f_E; and (iv) independent of an equal (additive) increase in both firms learning functions.*

The result in (i) was already noted earlier and is entirely intuitive—being higher up its learning curve gives I more of an advantage. It implies that when both firms face the same learning curve but I starts with an advantage of having sold to more consumers previously, I's initial competitive advantage is increasing in this difference in past sales.

The results in (ii) and (iii) follow directly from an inspection of (1). Simply put, the higher a firm's learning threshold or the value the firm gains from learning, the bigger that firm's competitive advantage. Adding the same constant to both f_I and f_E does not change $\Delta(N_I, N_E)$, so any additive change to the value of learning that affects both firms equally cancels out, which explains the result in (iv).

To explore the implications of changes that affect both firms at the same time, we focus on specific functional forms. We first consider a case with asymmetric linear learning functions in which both firms are an equal number of periods away from reaching their maximum learning threshold.

Proposition 4. *Consider different linear learning functions $f_i(x) = \theta_i x$ for $i = I, E$, so the cutoff is given by (4). When both firms are m periods away from their respective thresholds, I's competitive advantage is (i) decreasing in m iff $\theta_I > \theta_E$; (ii) increasing in an equal additive increase in $\overline{N}_I$ and $\overline{N}_E$ iff $\theta_I > \theta_E$; (iii) increasing in an equal additive increase in θ_I and θ_E iff $\overline{N}_I > \overline{N}_E$; and (iv) provided $m \geq 1$, increasing in δ iff $\theta_I > \theta_E$.*

The result in (i) means that if it takes longer to reach the maximum learning threshold (i.e. a shift back along the existing linear learning curve), then this helps the firm that adds less value from learning each period. This is because its learning disadvantage is temporarily reset back to a lower level. The result in (ii) reflects that when the maximum amount of learning that can be achieved is higher, then the firm with a higher marginal value of learning will benefit more. For a similar reason, if the marginal value of learning uniformly increases across the two firms, this benefits the firm that enjoys a higher learning curve, as shown in (iii). Finally, (iv) implies that as firms become more patient, this benefits the firm that has a higher marginal value of learning.

Next we consider a different form of asymmetry between the two firms. Specifically, we now allow I to be further along its learning curve than E (i.e. closer to the maximum learning threshold) while assuming both firms face identical learning curves. In such a setting, we can obtain results with a more general learning curve, specifically, the general S-shaped learning curve of Example 3.

Proposition 5. *Consider identical learning functions $f \equiv f_I = f_E$ and $\overline{N} \equiv \overline{N}_I = \overline{N}_E$ that take the form of the general S-shaped learning function of Example 3, with the cutoff $\Delta(N_I, N_E)$ given by (7). Suppose $N_I > N_E$ (which captures I's initial advantage). Then I's competitive advantage is (i) increasing in the marginal value of learning (i.e. θ); (ii) increasing in how close both firms are to obtaining some positive value from learning (i.e. higher when both N_I and N_E increase by k, provided $N_I + k \leq \underline{N}$ and $N_E + k \leq \underline{N}$); (iii) decreasing in how close both firms are to obtaining maximum learning given they already have obtained some positive value from learning (i.e. lower when both firms are k periods closer* *to the maximum learning threshold provided $N_I > \underline{N}$ and $N_E > \underline{N}$); and (iv) decreasing in δ if both firms are on the increasing part of the learning curve (i.e. provided $N_I > \underline{N}$ and $N_E > \underline{N}$).*

The result in (i) follows from the fact that $\Delta(N_I, N_E) > \Delta(N_E, N_E) = 0$ when $N_I > N_E$. This means $\Delta(N_I, N_E)$ must be increasing in θ: the more value is created by learning, the more I's initial advantage is amplified.

The result in (ii) is less obvious. It can be interpreted as considering what happens when both firms experience some improvement in their learning (e.g. access to some new data or some new technology that allows them to speed up their initial learning) so that both firms find themselves further along the initial flat part of their learning curve (i.e. closer to where they can offer positive value from learning). Alternatively, it can be interpreted as considering what happens when the learning technology improves so firms require fewer periods of initial consumer data until the value from learning starts to increase. Either way, the result in (ii) implies that this common improvement in the learning technology increases I's competitive advantage. To understand this result note that if firms are yet to move onto the increasing part of the learning curve, requiring fewer periods of customer data to reach the increasing part brings forward (i.e. discounts less) the advantage that I obtains from being closer to the threshold, thereby increasing its competitive advantage.

This result reverses in (iii), when firms are already on the increasing part of the learning curve. Here, a reduction in the number of learning periods both firms need to reach the maximum threshold decreases the number of periods over which I can enjoy its learning advantage, so helps E.[4] This result is similar to (i) in Proposition 4.

Finally, the result in (iv) signs the effect of firms becoming more patient (i.e. an increase in δ) on I's competitive advantage. This captures the same mechanism as in (iii), since making both firms more patient has a similar effect to moving them both closer to the threshold along the increasing part of the learning curve.

4 Within-user learning

So far we have considered pure across-user learning. In this section we focus on pure within-user learning instead. To make the model directly comparable to the case with across-user learning, we normalize the measure of consumers each period to one. In order to

[4]Other cases which involve firms moving from the initial flat part of the learning curve to the increasing part of the learning, or firms being on different parts of the learning curve, are ambiguous, reflecting the effects in (ii) and (iii) work in opposite directions.

focus on pure within-user learning, we assume these are the same consumers in every period. Specifically, consumers are infinitely lived but can switch firms in any period.

The value to a consumer of firm i's product with standalone value s_i in a given period is $s_i + f_i\left(\min\left\{N_i, \overline{N}_i\right\}\right)$, as before. The difference is that here N_i refers to the number of times the consumer has purchased from the same firm before, rather than the total number of consumers that the firm has sold to. Another key difference that naturally arises in the case of within-period learning is the possibility for firms to price discriminate based on the individual consumer's history since unlike the case of across-user learning, here the individual consumer's history affects the value offered by each firm's product. This is also consistent with the perfect information setup, so both firms can observe how many periods a customer has consumed from each firm previously, and can price discriminate accordingly.[5] The fact that there is no across-user learning and that firms can price discriminate with respect to individual consumers ensures that the analysis can be done for each individual consumer separately.

Then we can characterize when E wins, when I wins, and the value obtained by firms and consumers.

Proposition 6. *Suppose I has sold N_I times in the past to a given customer and E has sold N_E times to that customer, where $0 \leq N_I \leq \overline{N}_I$ and $0 \leq N_E \leq \overline{N}_E$. Then a unique MPE exists in which E wins in all periods iff $s_E - s_I \geq \Delta(N_I, N_E)$, and I wins in all periods otherwise, where $\Delta(N_I, N_E)$ is the same as in the case with across-user learning, i.e. $\Delta(N_I, N_E)$ is defined by (1). The outcome is socially optimal. Moreover, the firms' value functions are determined as follows:*

- *if $s_E - s_I < \Delta(N_I, N_E)$, then $V^E(N_I, N_E) = 0$ and $V^I(N_I, N_E) = \frac{s_I - s_E + \Delta(N_I, N_E)}{1-\delta}$ (with subsidization by E iff $\Delta(N_I, N_E + 1) \leq s_E - s_I < \Delta(N_I, N_E)$)*
- *if $s_E - s_I \geq \Delta(N_I, N_E)$, then $V^E(N_I, N_E) = \frac{s_E - s_I - \Delta(N_I, N_E)}{1-\delta}$ and $V^I(N_I, N_E) = 0$ (with subsidization by I iff $\Delta(N_I, N_E) \leq s_E - s_I < \Delta(N_I + 1, N_E)$),*

where we have once again used the convention $\Delta(\overline{N}_I + 1, N_E) \equiv \Delta(\overline{N}_I, N_E)$ for any $0 \leq N_E \leq \overline{N}_E$ and $\Delta(N_I, \overline{N}_E + 1) \equiv \Delta(N_I, \overline{N}_E)$ for any $0 \leq N_I \leq \overline{N}_I$.

Proposition 6 shows that the cutoff in $s_E - s_I$ that determines which firm wins in each period coincides with that obtained in the pure across-user case (with Pareto beliefs), and

[5] Obviously, each firm will know how many times a consumer has purchased from it before, so can condition its prices on this information. Given the model setup in which the same consumers are present in every period, each firm would therefore also be able to work out how many times a consumer has purchased from the rival firm in the past. In practice, firms may elicit this information from consumers by requiring a consumer supply their past purchase records so as to provide them with a better deal if they switch.

therefore remains socially optimal. As in the case with across-user learning, all the results in the Proposition remain valid when $\overline{N}_I \to \infty$ and $\overline{N}_E \to \infty$, provided learning functions are bounded by some power function. This means the comparative static results in Section 3.2 carry over to the case of within-user learning. However, despite this equivalence, the logic behind the result and its proof are quite different. With across-user learning, we never had to solve the consumers' forward-looking problem given their choice in any given period didn't affect their options in subsequent periods. In contrast, here consumers have to take into account the effect of their current choice of firm on their future options, and so we need to determine the value functions for consumers as well as for firms. In the proof of Proposition 1, we construct the consumers' value functions, and show that consumers never expect to switch firms in equilibrium.

In contrast to the case of across-user learning, the logic for why the equilibrium outcome under within-user learning coincides with the socially optimal outcome is quite straightforward. Here consumers are forward looking, so they take into account all future subsidies and utility they will receive from their chosen firm. Starting from any point, whichever firm creates greater PDV will be able to offer a higher level of utility plus subsidy to consumers, and so will be able to win in the current period. And if a particular firm wins in the current period, this just further increases the value it can offer in future periods, thereby ensuring that the firm can win in all future periods. Further reinforcing this intuition, in the Online Appendix we confirm that the outcome with pure within-user learning and a finite time horizon is also always socially optimal (recall that with pure across-user learning, the outcome with a finite time horizon was generally not socially optimal).

To further highlight the fact that although the cutoff is the same, the rationale for the result is different, note that prices no longer drive the equilibrium result. Indeed, if prices were set to zero in all periods (which rules out any subsidies), consumers would simply compare their respective present discounted willingness-to-pay between the two firms, so the condition for E to win would remain the same. This is in contrast to the case with learning across users, where removing the possibility of subsidization substantially changed the equilibrium condition for E to win.

Another key difference relative to the results with across-user learning is that, although the cutoff for E to win is the same, the payoffs to consumers and firms are different. Interestingly, the winning firm always does better with across-user learning than with the equivalent setting under within-user learning.

Proposition 7. *Comparing the case of within-user learning with across-user learning for the same learning curves, the same learning thresholds, and the same current state* (N_I, N_E)*, then provided either* $N_I < \overline{N}_I$ *or* $N_E < \overline{N}_E$*, the PDV of the winning firm's profit is strictly* *higher under across-user learning than in the case of within-user learning.*

The reason behind the result in Proposition 7 is that the winning firm needs to price more aggressively under within-user learning because consumers are forward looking. The logic is that under within-user learning consumers take into account that the surplus they will be left with in any period is the surplus offered by the losing firm (due to Bertrand competition), and that this surplus will be lower if they buy from the winning firm this period versus if they buy from the losing firm this period. Thus, the winning firm must compensate consumers by pricing lower in the current period compared to the case with across-user learning, in which consumers did not need to take into account future utility when making their choice of firm. Indeed, with across-user learning, each individual consumer's choice in the current period had no impact on the relative competitiveness of the two firms next period (since consumers are atomistic). Obviously, Proposition 7 implies that consumers are better off under within-user learning than under the comparable situation with across-user learning. This result can also be interpreted as meaning that forward-looking consumers are better off than myopic consumers. Indeed, the setting with pure within-user learning but myopic consumers would result in exactly the same outcome as the case with across-user learning, where consumers had no reason to consider more than their current surplus when deciding which firm to buy from.

5 Endogenous network effects and consumer beliefs

So far network effects have not played any role in the equilibrium analysis. In particular, consumers have not had to form beliefs about which firm other consumers will coordinate on in any given period. In Section 5.1 this was because learning was specific to each consumer and firms could price discriminate with respect to individual consumers. With across-user learning in Section 3, although there was a kind of self-reinforcing cycle across periods in which a firm that attracts consumers this period improves its product and so finds it easier to attract consumers in subsequent periods, current consumers still did not face the usual coordination problem that arises under classic network effects. This reflected that even when consumers were long-lived, each consumer was atomistic so their decision had no effect on the opportunities they could enjoy in subsequent periods, and they could costlessly switch each period to the best possible offer.

The purpose of this section is to show that network effects which create a coordination problem for consumers can arise in our framework. We will consider three ways in which this can happen. In Section 5.1 we show this happens once within-period learning is added to the pure across-user learning from Section 3. Then consumers need to form expectations on how many other consumers will choose the same product in any given period since this affects the value they get from the product in that period. In Section 5.2 we show such network effects arise endogenously when consumers only have to pay once to purchase a given firm's product or service, and then derive utility for multiple periods. In this case, consumers have to anticipate the learning that will occur in the future periods during which they derive utility, since the price paid creates an endogenous switching cost. This means each consumer has to form expectations over how many other consumers adopt the same product in the current period and in the future. Finally, in Section 5.3 we show network effects matter even in the absence of any exogenous switching cost or price (the products are assumed to be available for free) once we combine across-user and within-user learning.

5.1 Across-user learning and within-period learning

Consider our benchmark model of Section 3 in which there is across-user learning, but now allow firms to learn immediately from consumers purchasing in the current period, i.e. we allow for "within-period" learning. As argued in the introduction, within-period learning is a key distinguishing feature of cloud-based learning from customers, and contrasts with earlier types of learning from customers, where the new product features were only embedded in the next releases of the product which would be sold to *future* consumers. Within-period learning is assumed to work in exactly the same way as learning from past sales to consumers: it is just like another period of learning, except it provides value to current consumers. Specifically, if firm i has previously sold to a total measure of N_i consumers and sells to $n_i \leq 1$ consumers in the current period, then the value current consumers obtain from firm i's product is $s_i + f_i\left(\min\left\{N_i + n_i, \overline{N}_i\right\}\right)$, where $\overline{N}_i$ is the number of periods of sales (including the current period) it takes for firm i to reach the maximum threshold of learning.

A consequence of combining within-period learning and across-user learning is that there can be multiple equilibria in consumers' decision of which firm to buy from for given prices in any period, just like with regular network effects. Specifically, consumers may prefer buying from I if they expect all other consumers to buy from I in the current period, and they may prefer to buy from E if they expect all other consumers to buy from E in the current period. We will compare the outcomes under two types of beliefs. First, we assume that consumers believe all other consumers will coordinate on the equilibrium outcome that is best for them in the given period. This selects the equilibrium which maximizes the consumers' surplus (i.e. net utility) in that period. In the case consumers live for one period only, which recall is one interpretation of our across-user learning setup, this will indeed correspond to the

Pareto optimal equilibrium for consumers, and with this in mind, we refer to these beliefs as "Pareto" beliefs. We will then contrast these beliefs with beliefs that always favor the incumbent. This means that in each period consumers believe all other consumers will buy from I whenever for given prices, buying from I is an equilibrium outcome for consumers.

To see the difference between Pareto beliefs and beliefs favoring I more clearly, suppose I has learned from N_I previous consumers and charges p^I in the current period, while E has learned from N_E previous consumers and charges p^E in the current period. Then under Pareto beliefs, current consumers coordinate on buying from E if and only if

$$s_E + f_E\left(N_E + 1\right) - p^E \geq s_I + f_I\left(N_I + 1\right) - p^I$$

and they coordinate on buying from I otherwise. In contrast, under beliefs favoring I, current consumers coordinate on buying from E if and only if

$$s_E + f_E\left(N_E\right) - p^E \geq s_I + f_I\left(N_I + 1\right) - p^I$$

and they coordinate on buying from I otherwise. Other than that, the analysis for both cases proceeds in the same way as the baseline analysis in Section 3. As a result, relative to Proposition 1, the case of within-period learning and Pareto beliefs is as if both firms were one period closer to their respective maximum learning thresholds, whereas the case of within-period learning and beliefs favoring I is as if only I were one period closer to its maximum learning threshold. Then following an almost identical proof to that of Proposition 1 we obtain the following result.

Proposition 8. *Suppose I has previously sold to* $0 \leq N_I \leq \overline{N}_I$ *consumers and E has previously sold to* $0 \leq N_E \leq \overline{N}_E$ *consumers. Under Pareto beliefs, E wins in all periods if and only if* $s_E - s_I \geq \Delta(N_I + 1, N_E + 1)$*, and I wins in all periods otherwise, where* $\Delta(N_I, N_E)$ *is defined by (1). Under favorable beliefs for I, E wins in all periods if and only if* $s_E - s_I \geq \Delta(N_I + 1, N_E)$*, and I wins in all periods otherwise. The equilibrium outcome is socially efficient under Pareto beliefs and involves I winning too often under beliefs favoring I.*

The intuition for these results is the same as before. Note the socially optimal outcome takes into account the higher willingness-to-pay created from learning within the period, just as the equilibrium conditions do. Moreover, Corollary 1 and Corollary 2 continue to apply, given that the cutoffs that apply here are defined in the same way as (1), but with the terms in N_I and N_E increased by zero or one. For the same reason, the comparative static results also all carry over.

Taking the difference in cutoffs when beliefs favor I vs. when beliefs are Pareto, we obtain

$$\Delta(N_I+1, N_E) - \Delta(N_I+1, N_E+1) = (1-\delta)\left(\sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j \left(f_E(N_E+j+1) - f_E(N_E+j)\right)\right), \tag{8}$$

which is strictly positive whenever $N_E < \overline{N}_E$.[6] The expression in (8) is a measure of the distortion that arises in the cutoff for E to win due to favorable beliefs, as compared to the socially optimal outcome. It also shows that whether I is close or far away from its threshold is irrelevant to the distortion implied by I enjoying favorable beliefs. The expression (8) also illustrates that the impact of beliefs on I's competitive advantage reflects differences in the value of E's product from one more period of learning (under Pareto beliefs) across all future periods, rather than just one particular period. Taking the difference between (8) evaluated at $N_E - k$ and (8) evaluated at N_E, the following result follows almost immediately.

Proposition 9. *Suppose there is pure across-user learning and within-period learning. If $f_E(x)$ is weakly concave in x, the distortion in the cutoff resulting from beliefs favoring I is higher the further away E is from its maximum learning threshold (i.e. the lower N_E is).*

Provided f_E is weakly concave, the distortion in the cutoff for E due to beliefs favoring I is higher when E is on the earlier (steeper) part of its learning curve as opposed to the latter (flatter) part of its learning curve. This reflects that shifting back E by one additional period of learning has a bigger effect when E is on the steeper part of its learning function, and that the corresponding disadvantage for E arises over additional periods when E is further away from its maximum learning threshold. Put more directly, under the conditions of Proposition 9, the concern that I's competitive advantage from data gives it an inefficient advantage is greater the further away E is from its maximum learning threshold, but does not depend on how close I is to its maximum learning threshold.

5.2 Across-user learning with one-time purchase

Consider the model of pure across-user learning from Section 3 but now instead of allowing for within-period learning as we did in Section 5.1, assume consumers only need to purchase once to derive consumption utility over all subsequent periods based on the value of the product offered in each period. Thus, we have in mind products that can be used over multiple periods of time, e.g. operating systems, software, apps and connected devices that are purchased upfront and can be used repeatedly at no additional cost thereafter. We assume

[6] When $N_E = \overline{N}_E$, beliefs are irrelevant, and (8) is zero.

there is a measure one of new consumers joining in each period and all consumers are infinitely lived. These consumers purchase one of the two firms' products and get to enjoy the stream of future utility associated with their choice of product. Consumers could, if they wanted, switch to use the other product instead, at any point, but this would require purchasing that product as well. Finally, consistent with our previous model of across-user learning in case consumers are new in each period, a firm only learns once from any consumers it sells to. Formally, if firm i has sold to a total measure of N_i consumers in the past then the value current consumers obtain from firm i's product is $s_i + f_i\left(\min\left\{N_i, \overline{N}_i\right\}\right)$, where $\overline{N}_i$ is the total measure of unique past sales it takes for firm i to reach the maximum threshold of learning.

Once again, there can be multiple equilibria in consumers' decision of which firm to buy from for given prices in any period. To understand why network effects matter for consumers here despite the absence of any within-period learning, note that due to the repeated usage consumers enjoy from a purchase, consumers face a type of switching cost: if a consumer buys a product and then decides to switch to the other product in a later period, she needs to purchase the other product, whereas she could have continued to consume the existing product without any further payment. This means that, all other things equal, consumers prefer to buy the product which they expect will also be bought by current and future consumer cohorts, because that product will benefit from the most learning in the future. Because of this switching cost, each consumer now has an incentive to coordinate her adoption decision with the adoption decisions of the other consumers arriving in the same period in order to choose the product that will benefit from the most learning in the future.

Beliefs once again play an important role and, like in the previous subsection, we highlight their effect by comparing the outcomes with Pareto beliefs vs. beliefs that always favor I. To see where the difference between the two types of beliefs comes from more clearly, suppose I has learned from N_I previous consumers and charges p^I in the current period, while E has learned from N_E previous consumers and charges p^E in the current period. Then under Pareto beliefs, current consumers coordinate on buying from E if and only if

$$\begin{aligned} &\sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j \left(s_E + f_E\left(N_E + j\right)\right) + \delta^{\overline{N}_E - N_E}\left(\frac{s_E + f_E\left(\overline{N}_E\right)}{1-\delta}\right) - p^E \\ \geq\; &\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j \left(s_I + f_I\left(N_I + j\right)\right) + \delta^{\overline{N}_I - N_I}\left(\frac{s_I + f_I\left(\overline{N}_I\right)}{1-\delta}\right) - p^I. \end{aligned}$$

Otherwise, they coordinate on buying from I. In contrast, under beliefs favoring I, current consumers coordinate on buying from E if and only if

$$\begin{aligned}&\frac{s_E + f_E\left(N_E\right)}{1-\delta} - p^E \\ \geq\ & \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j \left(s_I + f_I\left(N_I + j\right)\right) + \delta^{\overline{N}_I - N_I} \left(\frac{s_I + f_I\left(\overline{N}_I\right)}{1-\delta}\right) - p^I.\end{aligned}$$

Otherwise, they coordinate on buying from I. Thus, with Pareto beliefs, consumers evaluate each firm assuming all future cohorts of consumers join that firm, so the surplus it offers in each period increases over time. With beliefs favoring I, consumers evaluate E assuming all future cohorts of consumers join I, so the surplus offered by E in each period remains stuck at $s_E + f_E\left(N_E\right)$. Other than that, the analysis proceeds in the exact same way in both cases.

The complete proof, which is lengthy, follows very similar steps as the proof of Proposition 1.

We relegate it to the Online Appendix, where we establish the following proposition.

Proposition 10. *Suppose I has previously sold to* $0 \leq N_I \leq \overline{N}_I$ *unique consumers and E has previously sold to* $0 \leq N_E \leq \overline{N}_E$ *unique consumers. Under Pareto beliefs, E wins iff* $s_E - s_I \geq \Delta(N_I, N_E)$, *where*

$$\begin{aligned}\Delta(N_I, N_E) \ \equiv\ & (1-\delta)^2 \left(\sum_{j=0}^{\overline{N}_I - N_I - 1} (j+1)\delta^j f_I(N_I + j) - \sum_{j=0}^{\overline{N}_E - N_E - 1} (j+1)\delta^j f_E(N_E + j)\right) \\ & + \delta^{\overline{N}_I - N_I}(1 + (1-\delta)(\overline{N}_I - N_I)) f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E}(1 + (1-\delta)(\overline{N}_E - N_E)) f_E(\overline{N}_E).\end{aligned}$$

In contrast, under favorable beliefs for I, E wins iff $s_E - s_I \geq \Delta^I(N_I, N_E)$, *where*

$$\begin{aligned}\Delta^I(N_I, N_E) \ =\ & (1-\delta)^2 \left(\sum_{j=0}^{\overline{N}_I - N_I - 1} (j+1)\delta^j f_I(N_I + j) - \sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j)\right) \\ & + \delta^{\overline{N}_I - N_I}(1 + (1-\delta)(\overline{N}_I - N_I)) f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E).\end{aligned}$$

The cutoff in $s_E - s_I$ *above which E wins with Pareto beliefs is socially efficient and lower than the cutoff in* $s_E - s_I$ *above which E wins with beliefs favorable to I, i.e.* $\Delta(N_I, N_E) < \Delta^I(N_I, N_E)$ *for all* (N_I, N_E) *such that* $0 \leq N_I \leq \overline{N}_I$ *and* $0 \leq N_E \leq \overline{N}_E - 1$.

To understand this result, note that in each state, consumers evaluate the firms based on the utility they offer in the current period (which reflects the firms' additional learning), plus the subsidy each firm offers in the current period (reflecting how much each firm stands to benefit if it is able to capture consumers in the current period relative to the case when it loses the current period), plus the PDV of the future utility stream that the consumer expects will be offered in all subsequent periods. The firms' value functions are defined in the proof of Proposition 10. They turn out to be identical to the expressions in Corollary 1 except that the $\Delta(N_I, N_E)$ is now defined as above (and depends on whether beliefs are Pareto or favor I), and that we need to divide everywhere by an additional $1 - \delta$ in order to reflect that consumers get an infinite stream of future benefits from the product when they purchase.

The difference between the two types of beliefs is in the value consumers expect E to offer in subsequent periods: with favorable beliefs for I, when E has previously sold to N_E unique consumers, the relevant expected value offered by E in all subsequent periods is stuck at $f_E(N_E)$, rather than increasing to $f_E(N_E + 1)$, $f_E(N_E + 2)$, etc., and eventually reaching $f_E(\overline{N}_E)$. This shows that beliefs clearly affect E's ability to win: E wins more often under Pareto beliefs relative to favorable beliefs for I. Moreover, favorable beliefs for I lead to a socially inefficient cutoff rule, in which I sometimes wins even though it would be efficient for consumers to buy from E.

5.3 Across-user and within-user learning

In this final section we demonstrate that combining across-user and within-user learning can also generate network effects that matter for consumers. To show this we shut down the channels that led to such network effects in the previous two sections—within-period learning (Section 5.1), and switching costs that arise via prices (Section 5.2). We do so by combining the models of Sections 3 and 4 in the following way. First, we assume that the value a given consumer obtains from firm i's product in the current period depends on two types of learning—the total measure of consumers who have previously purchased from the same firm, and the number of times the same consumer has previously purchased from the same firm. I.e. we allow for both across-user and within-user learning. Second, we set prices of both firms to zero in all periods to show that prices are not necessary to obtain our result (meaning our theory can also apply to free products, such as Google search). And finally, we assume there is just one fixed set of consumers who are present in all periods, consistent with the model of within-user learning used in Section 4.

Formally, we assume there is a measure one of consumers that are present in every period. They have to decide which firm to use each period. While they can only consume from one firm in a given period, they are free to switch between the two firms between periods given prices are equal to zero in all periods. Consumers get utility from each period's consumption. Specifically, a consumer that has previously bought n_i times from firm $i \in \{I, E\}$ when firm i has previously served $N_i \leq 1$ of all consumers in the past derives utility $s_i + f^A(N_i) + f^W(\min\{n_i, 1\})$ in the current period if it chooses firm i's product. We assume the learning functions f^A and f^W are strictly increasing and $f^A(0) = f^W(0) = 0$. Thus, in order to focus on the simplest possible case that combines both types of learning, we have assumed the two firms have the same learning functions and the maximum utility that can ever be offered by firm i to an individual consumer in a period is $s_i + f^A(1) + f^W(1)$, which is achieved when all consumers have bought at least once from firm i in the past.[7]

Here, a coordination problem arises because of the endogenous switching cost created by within-user learning, which in turn implies that each consumer wants to coordinate on the firm that will attract the other consumers, in order to also benefit from across-user learning. Of course, when both firms have reached maximum learning (i.e. when both have served all consumers at least once in the past so they offer utility $s_I + f^A(1) + f^W(1)$ and $s_E + f^A(1) + f^W(1)$, respectively), there is no coordination problem: the unique equilibrium has consumers choosing I in every period if $s_I > s_E$, and otherwise choosing E in every period. Now suppose only one of the firms, say I, has reached its maximum threshold of learning, so offers utility $s_I + f^A(1) + f^W(1)$ in every period, whereas E has not had learning in the past, so offers only s_E in the current period. One might expect a coordination problem would be at play here. However, as we show in the proof of Proposition 11, it turns out that once again the equilibrium is unique and thus does not depend on consumer beliefs: consumers choose I in every period if $s_I + f^A(1) + f^W(1) > s_E + \delta f^W(1)$, and otherwise they choose E in every period. This equilibrium is inefficient, in that I wins too often (in the efficient outcome, consumers would choose I in every period if and only if $s_I + f^A(1) + f^W(1) > s_E + \delta\left(f^A(1) + f^W(1)\right)$). This inefficiency reflects E's inability to offer a subsidy to get current consumers to internalize the future benefit generated via across-user learning when they choose E (note they do individually internalize the future benefit of within-user learning).

Thus, in order to clearly illustrate how network effects create a coordination problem for consumers in this setting, we focus on the case in which both firms start with a blank state, that is, $N_I = N_E = 0$ and $n_I = n_E = 0$.

Proposition 11. *Suppose both firms start with no previous learning, i.e.* $N_I = N_E = 0$ *and* $n_I = n_E = 0$ *for every consumer. Under Pareto beliefs, E wins if and only if* $s_E - s_I \geq 0$, *which is the socially optimal threshold given the two firms are symmetric. Under favorable beliefs for I, E wins if and only if* $s_E - s_I \geq \delta \min\left\{f^W(1), f^A(1)\right\}$, *i.e. the threshold for E*

[7] In the Online Appendix we provide the analysis for the more general case, in which learning functions can be different.

to win is too high in this case compared to the socially optimal threshold.

Like in the two previous sections, beliefs that favor I shift the outcome in I's favor relative to the case with Pareto beliefs, which remains socially optimal. Indeed, consumers who expect other consumers to choose I in the current period will expect less utility from choosing E compared to the case with Pareto beliefs, and therefore will choose I even if they may have chosen E under Pareto beliefs. This shows that network effects and beliefs matter when both across-user and within-user learning are at play, even though consumers incur no switching costs and do not have to pay a price to purchase from either firm.

It is important to note that both types of learning are necessary for network effects to matter in this model. If either type of learning is shut down (i.e. if either $f^W(1) = 0$ or $f^A(1) = 0$), then the condition for E to win under beliefs favorable to I becomes identical to the condition for E to win under Pareto beliefs (which is socially efficient), so the nature of beliefs and network effects no longer matter. Despite the apparent symmetry, the two types of learning actually play different roles. If we shut down across-user learning, i.e. if $f^A(1) = 0$, then network effects disappear because each consumer only cares about their own individual utility, so they compare $\frac{s_E+\delta f^W(1)}{1-\delta}$ with $\frac{s_E+\delta f^W(1)}{1-\delta}$, i.e. s_E with s_I (this is a simplified version of our model with pure within-user learning from Section 4). If instead we shut down within-user learning, i.e. if $f^W(1) = 0$, then switching costs disappear because a consumer's individual choice in one period no longer has any effect on their options in subsequent periods. This means that in every period consumers only consider the current utility offered by each firm, which only depends on past learning, so E wins if and only if $s_E \geq s_I$ (this is a simplified version of our model with pure across-user learning from Section 3).

6 Conclusion

Data-enabled learning is becoming an essential ingredient to the competitive dynamics between providers of an increasing range of products and services. We have analyzed three key determinants of competitive advantage in the presence of data-enabled learning: (i) the shape of each firm's learning curve and the competitors' relative positions on their respective learning curves, (ii) the relative importance of across-user vs. within-user learning, and (iii) user beliefs whenever data-enabled learning creates a user coordination problem due to network effects.

Despite allowing for very general learning functions and an infinite period dynamic model of competition, the framework we have developed is highly tractable. This makes it suitable for many interesting extensions. Some interesting extensions to explore are (i) generalizing our analysis with across-user to allow for a new set of infinitely lived consumers joining each period, (ii) generalizing our model of combined across-user and within-user learning to allow for new consumers to join each period, (iii) allowing for depreciation of old data over time; (iv) and exploring the implications of data sharing under different settings.

7 Appendix

This appendix contains the proofs of results not already proven in the main text.

7.1 Proof of Proposition 1 and Corollary 1

The state $(N_I, N_E) = (\overline{N}_I, \overline{N}_E)$ is already handled in the main text. Recall the value function of profit in this case is given by

$$V^i(\overline{N}_I, \overline{N}_E) = \frac{1}{1-\delta} \max\left\{s_i - s_j + f_i(\overline{N}_i) - f_j(\overline{N}_j), 0\right\}.$$

Consider the case with $N_I = \overline{N}_I$, and suppose E has previously sold to N_E consumers, where $1 \leq N_E \leq \overline{N}_E$. Suppose

$$\Delta(\overline{N}_I, N_E) = f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E + 1} f_E(\overline{N}_E) - (1-\delta) \sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E + j)$$

and

$$V^I(\overline{N}_I, N_E) = \begin{cases} \frac{s_I - s_E + f_I(\overline{N}_I) - f_E(N_E)}{1-\delta} & \text{if} \quad s_E - s_I < \Delta(\overline{N}_I, N_E + 1) \\ \frac{s_I - s_E + \Delta(\overline{N}_I, N_E)}{(1-\delta)^2} & \text{if} \quad \Delta(\overline{N}_I, N_E + 1) \leq s_E - s_I < \Delta(\overline{N}_I, N_E) \\ 0 & \text{if} \quad s_E - s_I \geq \Delta(\overline{N}_I, N_E) \end{cases}$$

$$V^E(\overline{N}_I, N_E) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta(\overline{N}_I, N_E) \\ \frac{s_E - s_I - \Delta(\overline{N}_I, N_E)}{1-\delta} & \text{if} \quad s_E - s_I \geq \Delta(\overline{N}_I, N_E) \end{cases}.$$

Now suppose E has previously sold to $N_E - 1$ consumers. Suppose E charges p^E and I charges p^I in the current period. I wins the current period if $s_I + f_I(\overline{N}_I) - p^I > s_E + f_E(N_E - 1) - p^E$. Otherwise, E wins. If E wins the current period, the PDV of its current and future profits will be $p^E - c + \delta V^E(\overline{N}_I, N_E)$. If it doesn't win the current period, the PDV of its current and future profits will be $\delta V^E(\overline{N}_I, N_E - 1)$. So E is willing to price down to the point where $p^E - c + \delta V^E(\overline{N}_I, N_E) = \delta V^E(\overline{N}_I, N_E - 1)$, or in other words $p^E = c - \delta(V^E(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1))$. Based on the same logic, I is willing to price down to $p^I = c - \delta(V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E))$. Thus, given

Bertrand competition, I wins the current period if and only if

$$\begin{aligned} & s_I + f_I(\overline{N}_I) + \delta(V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E)) \\ > \; & s_E + f_E(N_E - 1) + \delta(V^E(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1)). \end{aligned} \tag{9}$$

Suppose (9) holds. Then in the current period E sets $p^E = c - \delta(V^E(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1))$ and I wins by setting $p^I = c + s_I + f_I(\overline{N}_I) - s_E - f_E(N_E - 1) - \delta(V^E(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1))$. Then I's value is

$$\begin{aligned} V^I(\overline{N}_I, N_E - 1) &= s_I + f_I(\overline{N}_I) - s_E - f_E(N_E - 1) \\ &\quad - \delta(V^E(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1)) + \delta V^I(\overline{N}_I, N_E - 1), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} V^I(\overline{N}_I, N_E - 1) &= \delta V^I(\overline{N}_I, N_E) + s_I - s_E + f_I(\overline{N}_I) - f_E(N_E - 1) \\ &\quad + \delta(V^I(\overline{N}_I, N_E - 1) + V^E(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E)) \end{aligned}$$

and E's value is

$$V^E(\overline{N}_I, N_E - 1) = \delta V^E(\overline{N}_I, N_E - 1)$$

given it loses and so will be in the same situation next period. Since $\delta < 1$, this implies $V^E(\overline{N}_I, N_E - 1) = 0$.

Conversely, suppose (9) doesn't hold. Then in the current period I sets $p^I = c - \delta(V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E))$ and E wins by setting $p^E = c + s_E + f_E(N_E - 1) - s_I - f_I(\overline{N}_I) - \delta(V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E))$. Then E's value is

$$\begin{aligned} V^E(\overline{N}_I, N_E - 1) &= s_E + f_E(N_E - 1) - s_I - f_I(\overline{N}_I) \\ &\quad - \delta(V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E)) + \delta V^E(\overline{N}_I, N_E), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} V^E(\overline{N}_I, N_E - 1) &= \delta V^E(\overline{N}_I, N_E - 1) + s_E - s_I + f_E(N_E - 1) - f_I(\overline{N}_I) \\ &\quad + \delta(V^I(\overline{N}_I, N_E) + V^E(\overline{N}_I, N_E) - V^I(\overline{N}_I, N_E - 1) - V^E(\overline{N}_I, N_E - 1)) \end{aligned}$$

and I's value is

$$V^I(\overline{N}_I, N_E - 1) = \delta V^I(\overline{N}_I, N_E)$$

given it loses and will be in the state $(\overline{N}_I, N_E)$ next period.

Combining the above two cases, we have

$$V^I(\overline{N}_I, N_E - 1) = \delta V^I(\overline{N}_I, N_E) + \max \left\{ \begin{array}{c} s_I - s_E + f_I(\overline{N}_I) - f_E(N_E - 1) \\ +\delta \left(\begin{array}{c} V^I(\overline{N}_I, N_E - 1) + V^E(\overline{N}_I, N_E - 1) \\ -V^I(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E) \end{array} \right), 0 \end{array} \right\} \quad (10)$$

$$V^E(\overline{N}_I, N_E - 1) = \delta V^E(\overline{N}_I, N_E - 1) + \max \left\{ \begin{array}{c} s_E - s_I + f_E(N_E - 1) - f_I(\overline{N}_I) \\ +\delta \left(\begin{array}{c} V^I(\overline{N}_I, N_E) + V^E(\overline{N}_I, N_E) \\ -V^I(\overline{N}_I, N_E - 1) - V^E(\overline{N}_I, N_E - 1) \end{array} \right), 0 \end{array} \right\}. \quad (11)$$

There are two possibilities: $V^E(\overline{N}_I, N_E - 1) = 0$ and $V^E(\overline{N}_I, N_E - 1) > 0$.

Suppose first $V^E(\overline{N}_I, N_E - 1) = 0$. This implies $V^I(\overline{N}_I, N_E - 1) \geq \delta V^I(\overline{N}_I, N_E)$ and

$$\begin{aligned} V^I(\overline{N}_I, N_E - 1) &= \frac{s_I - s_E + f_I(\overline{N}_I) - f_E(N_E - 1) - \delta V^E(\overline{N}_I, N_E)}{1 - \delta} \\ &= \begin{cases} \frac{s_I - s_E + f_I(\overline{N}_I) - f_E(N_E - 1)}{1 - \delta} & \text{if} \quad s_E - s_I < \Delta(\overline{N}_I, N_E) \\ \frac{s_I - s_E + (1-\delta)(f_I(\overline{N}_I) - f_E(N_E - 1)) + \delta \Delta(\overline{N}_I, N_E)}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta(\overline{N}_I, N_E) \end{cases} \end{aligned}$$

Note that when $s_E - s_I < \Delta(\overline{N}_I, N_E)$, we have

$$\begin{aligned} V^I(\overline{N}_I, N_E - 1) &= \frac{s_I - s_E + f_I(\overline{N}_I) - f_E(N_E - 1)}{1 - \delta} \\ &\geq \frac{s_I - s_E + f_I(\overline{N}_I) - f_E(N_E)}{1 - \delta} \geq V^I(\overline{N}_I, N_E) > \delta V^I(\overline{N}_I, N_E). \end{aligned}$$

And when $s_E - s_I \geq \Delta(\overline{N}_I, N_E)$, we have $V^I(\overline{N}_I, N_E) = 0$, so the binding constraint on this range must be $V^I(\overline{N}_I, N_E - 1) \geq 0$, which is equivalent to

$$s_E - s_I \leq (1 - \delta)(f_I(\overline{N}_I) - f_E(N_E - 1)) + \delta \Delta(\overline{N}_I, N_E) = \Delta(\overline{N}_I, N_E - 1).$$

Next, suppose $V^E(\overline{N}_I, N_E - 1) > 0$, so $V^I(\overline{N}_I, N_E - 1) = \delta V^I(\overline{N}_I, N_E)$. Since $V^I\left(\overline{N}_I, N_E - 1\right) \geq V^I\left(\overline{N}_I, N_E\right)$, we must have $V^I(\overline{N}_I, N_E - 1) = V^I(\overline{N}_I, N_E) = 0$ in this case. So it must be that $s_E - s_I \geq \Delta(\overline{N}_I, N_E)$. In this range, we have

$$\begin{aligned} V^E(\overline{N}_I, N_E - 1) &= s_E - s_I + f_E(N_E - 1) - f_I(\overline{N}_I) + \delta V^E(\overline{N}_I, N_E) \\ &= \frac{s_E - s_I + (1 - \delta)(f_E(N_E - 1) - f_I(\overline{N}_I)) - \delta \Delta(\overline{N}_I, N_E)}{1 - \delta}, \end{aligned}$$

which is positive if and only if $s_E - s_I > \Delta(\overline{N}_I, N_E - 1)$.

Thus, we have proven that

$$V^I(\overline{N}_I, N_E-1) = \begin{cases} \frac{s_I-s_E+f_I(\overline{N}_I)-f_E(N_E-1)}{1-\delta} & \text{if} \quad s_E-s_I<\Delta(\overline{N}_I,N_E) \\ \frac{s_I-s_E+\Delta(\overline{N}_I,N_E-1)}{(1-\delta)^2} & \text{if} \quad \Delta(\overline{N}_I,N_E)\le s_E-s_I<\Delta(\overline{N}_I,N_E-1) \\ 0 & \text{if} \quad s_E-s_I\ge\Delta(\overline{N}_I,N_E-1) \end{cases}$$

$$V^E(\overline{N}_I, N_E-1) = \begin{cases} 0 & \text{if} \quad s_E-s_I<\Delta(\overline{N}_I,N_E-1) \\ \frac{s_E-s_I-\Delta(\overline{N}_I,N_E-1)}{1-\delta} & \text{if} \quad s_E-s_I\ge\Delta(\overline{N}_I,N_E-1) \end{cases}.$$

This establishes that the results in Proposition 1 and Corollary 1 hold for any $0 \le N_E \le \overline{N}_E$ when $N_I = \overline{N}_I$. Symmetry can be used to obtain a parallel result for all $0 \le N_I \le \overline{N}_I$ when $N_E = \overline{N}_E$, namely that

$$V^E(N_I,\overline{N}_E) = \begin{cases} 0 & \text{if} \quad s_E-s_I<\Delta(N_I,\overline{N}_E) \\ \frac{s_E-s_I-\Delta(N_I,\overline{N}_E)}{(1-\delta)^2} & \text{if} \quad \Delta(N_I,\overline{N}_E)\le s_E-s_I<\Delta(N_I+1,\overline{N}_E) \\ \frac{s_E-s_I+f_E(\overline{N}_E)-f_I(N_I)}{1-\delta} & \text{if} \quad s_E-s_I\ge\Delta(N_I+1,\overline{N}_E) \end{cases}$$

$$V^I(N_I,\overline{N}_E) = \begin{cases} \frac{s_I-s_E+\Delta(N_I,\overline{N}_E)}{1-\delta} & \text{if} \quad s_E-s_I<\Delta(N_I,\overline{N}_E) \\ 0 & \text{if} \quad s_E-s_I\ge\Delta(N_I,\overline{N}_E) \end{cases},$$

where

$$\Delta(N_I,\overline{N}_E) = \delta^{\overline{N}_I-N_I+1} f_I(\overline{N}_I) - f_E(\overline{N}_E) + (1-\delta)\sum_{j=0}^{\overline{N}_I-N_I} \delta^j f_I(N_I+j).$$

Now consider any state (N_I, N_E) with $0 \le N_I \le \overline{N}_I - 1$ and $0 \le N_E \le \overline{N}_E - 1$, and suppose that the result in Proposition 1 and Corollary 1 hold for the states (N_I+1, N_E) and (N_I, N_E+1), which means

$$V^I(N_I{+}1,N_E) = \begin{cases} \frac{s_I-s_E}{1-\delta} + \frac{\Delta(N_I+1,N_E)-\delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2} & \text{if} \quad s_E-s_I<\Delta(N_I+1,N_E+1) \\ \frac{s_I-s_E+\Delta(N_I+1,N_E)}{(1-\delta)^2} & \text{if} \quad \Delta(N_I+1,N_E+1)\le s_E-s_I\le\Delta(N_I+1,N_E) \\ 0 & \text{if} \quad s_E-s_I\ge\Delta(N_I+1,N_E) \end{cases}$$

$$V^E(N_I,N_E{+}1) = \begin{cases} 0 & \text{if} \quad s_E-s_I<\Delta(N_I,N_E+1) \\ \frac{s_E-s_I-\Delta(N_I,N_E+1)}{(1-\delta)^2} & \text{if} \quad \Delta(N_I,N_E+1)\le s_E-s_I\le\Delta(N_I+1,N_E+1) \\ \frac{s_E-s_I}{1-\delta} - \frac{\Delta(N_I,N_E+1)-\delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2} & \text{if} \quad s_E-s_I>\Delta(N_I+1,N_E+1) \end{cases}.$$

Using the same logic as above when the state is (N_I, N_E), we have

$$V^I(N_I,N_E) = \delta V^I(N_I,N_E+1) + \max\left\{ \begin{array}{c} s_I-s_E+f_I(N_I)-f_E(N_E) \\ +\delta\begin{pmatrix} V^I(N_I+1,N_E)+V^E(N_I+1,N_E) \\ -V^I(N_I,N_E+1)-V^E(N_I,N_E+1) \end{pmatrix}, 0 \end{array} \right\}$$

$$V^E(N_I,N_E)=\delta V^E(N_I+1,N_E)+\max\left\{\begin{array}{c} s_E-s_I+f_E(N_E)-f_I(N_I) \\ +\delta\left(\begin{array}{c} V^E(N_I,N_E+1)+V^I(N_I,N_E+1) \\ -V^E(N_I+1,N_E)-V^I(N_I+1,N_E)\end{array}\right),0\end{array}\right\}.$$

There are two possibilities. If

$$\max\left\{\begin{array}{c} s_I-s_E+f_I(N_I)-f_E(N_E) \\ +\delta(V^I(N_I+1,N_E)+V^E(N_I+1,N_E)-V^I(N_I,N_E+1)-V^E(N_I,N_E+1)),0\end{array}\right\}>0,$$

then $V^E(N_I,N_E)=\delta V^E(N_I+1,N_E)=0$, because we must have $V^E(N_I,N_E)\geq V^E(N_I+1,N_E)$. In this case, we have

$$V^I(N_I,N_E)=s_I-s_E+f_I(N_I)-f_E(N_E)+\delta(V^I(N_I+1,N_E)-V^E(N_I,N_E+1)).$$

On the other hand, if

$$\max\left\{\begin{array}{c} s_I-s_E+f_I(N_I)-f_E(N_E) \\ +\delta(V^I(N_I+1,N_E)+V^E(N_I+1,N_E)-V^I(N_I,N_E+1)-V^E(N_I,N_E+1)),0\end{array}\right\}=0,$$

then $V^I(N_I,N_E)=\delta V^I(N_I,N_E+1)=0$, because we must have $V^I(N_I,N_E)\geq V^I(N_I,N_E+1)$. In this case, we have

$$V^E(N_I,N_E)=s_E-s_I+f_E(N_E)-f_I(N_I)+\delta(V^E(N_I,N_E+1)-V^I(N_I+1,N_E)).$$

We have

$$s_I-s_E+f_I(N_I)-f_E(N_E)+\delta(V^I(N_I+1,N_E)-V^E(N_I,N_E+1))$$

$$=\begin{cases} \begin{array}{c} s_I-s_E+f_I(N_I)-f_E(N_E) \\ +\delta(\frac{s_I-s_E}{1-\delta}+\frac{\Delta(N_I+1,N_E)-\delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2})\end{array} & \text{if} \quad s_E-s_I<\Delta(N_I,N_E+1) \\ \\ \begin{array}{c} s_I-s_E+f_I(N_I)-f_E(N_E) \\ +\delta(\frac{s_I-s_E}{1-\delta}+\frac{\Delta(N_I+1,N_E)-\delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2})-\delta(\frac{s_E-s_I-\Delta(N_I,N_E+1)}{(1-\delta)^2})\end{array} & \text{if} \quad \begin{array}{c}\Delta(N_I,N_E+1)\leq s_E-s_I \\ \leq\Delta(N_I+1,N_E+1)\end{array} \\ \\ \begin{array}{c} s_I-s_E+f_I(N_I)-f_E(N_E) \\ +\delta(\frac{s_I-s_E+\Delta(N_I+1,N_E)}{(1-\delta)^2})-\delta(\frac{s_E-s_I}{1-\delta}-\frac{\Delta(N_I,N_E+1)-\delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2})\end{array} & \text{if} \quad \begin{array}{c}\Delta(N_I+1,N_E+1)\leq s_E-s_I \\ \leq\Delta(N_I+1,N_E)\end{array} \\ \\ \begin{array}{c} s_I-s_E+f_I(N_I)-f_E(N_E) \\ -\delta(\frac{s_E-s_I}{1-\delta}-\frac{\Delta(N_I,N_E+1)-\delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2})\end{array} & \text{if} \quad s_E-s_I\geq\Delta(N_I+1,N_E)\end{cases}$$

Straightforward calculations reveal that the expression in the first line is equal to

$$\frac{s_I - s_E}{1-\delta} + \frac{\Delta(N_I, N_E) - \delta\Delta(N_I, N_E+1)}{(1-\delta)^2},$$

while the expressions in the second and third lines are identical and equal to

$$\frac{s_I - s_E + \Delta(N_I, N_E)}{(1-\delta)^2}.$$

Since $s_I - s_E + f_I(N_I) - f_E(N_E) + \delta(V^I(N_I+1, N_E) - V^E(N_I, N_E+1))$ is continuous in $(s_I - s_E)$ and $\Delta(N_I, N_E) < \Delta(N_I+1, N_E)$, we can conclude that the expression of $V^I(N_I, N_E)$ given in Corollary 1 holds. By symmetry in I and E, the same is true for the expression of $V^E(N_I, N_E)$ given in Corollary 1. Thus, the induction hypothesis holds. By repeated application of the induction hypothesis, we obtain that the expressions for $V^I(N_I, N_E)$ and $V^E(N_I, N_E)$ in Corollary 1, and the expression for $\Delta(N_I, N_E)$ in Proposition 1 hold for all states (N_I, N_E), where $0 \le N_I \le \overline{N}_I - 1$ and $0 \le N_E \le \overline{N}_E - 1$.

Finally, we can compare $\Delta(N_I, N_E)$ with the socially optimal outcome. If I wins in every period, the value created is

$$\frac{s_I}{1-\delta} + \frac{\delta^{\overline{N}_I - N_I + 1}}{1-\delta} f_I(\overline{N}_I) + \sum_{j=0}^{\overline{N}_I - N_I} \delta^j f_I(N_I + j)$$

while if E wins in every period, the value created is

$$\frac{s_E}{1-\delta} + \frac{\delta^{\overline{N}_E - N_E + 1}}{1-\delta} f_E(\overline{N}_E) + \sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E + j).$$

The socially optimal cutoff is the level of $s_I - s_E$ for which these two expressions are equal, which corresponds exactly to the equilibrium cutoff $\Delta(N_I, N_E)$.

7.2 Proof of Proposition 2

To show Proposition 1 continues to apply in the limit as $\overline{N}_I \to \infty$ and $\overline{N}_E \to \infty$, we need to prove the expression in (1) continues to be well defined in this limit case. This is equivalent to proving $\lim_{N_i \to \infty} \delta^{N_i} f_i(N_i)$ and $\lim_{N_i \to \infty} \sum_{j=0}^{N_i} \delta^j f_i(j)$ exist. Consider the power function $f_i(x) = \theta_i x^{\rho_i}$ where $\theta_i > 0$ and $\rho_i > 0$. Using the ratio test, we know that $\lim_{j \to \infty}(\frac{\delta^{j+1}(j+1)^\rho}{\delta^j j^\rho}) = \delta < 1$ which implies both limits exist for any such power function. Then by the limit comparison test, since the relevant terms are all positive, the limits exists for any unbounded learning function f_i provided there exists some $\kappa_i > 0$, such that $\lim_{j \to \infty} \frac{f_i(j)}{j^\rho} \le \kappa_i$ for $i = \{I, E\}$.

7.3 Proof of Proposition 4

To prove (i), it can be shown that $\Delta(\overline{N}_I - (m+k), \overline{N}_E - (m+k)) - \Delta(\overline{N}_I - m, \overline{N}_E - m) = (\theta_I - \theta_E)\sum_{j=1}^{k}(\delta^{m+j} - 1) < 0$, so $\Delta(\overline{N}_I - (m+k), \overline{N}_E - (m+k)) < \Delta(\overline{N}_I - m, \overline{N}_E - m)$ iff $\theta_I > \theta_E$. To prove (ii) note if $\overline{N}_I$ and $\overline{N}_E$ increase by k, then the change in $\Delta(\overline{N}_I - m, \overline{N}_E - m)$ is $(\theta_I - \theta_E)k$. To prove (iii) note if θ_I and θ_E increase by k, then the change in $\Delta(\overline{N}_I - m, \overline{N}_E - m)$ is $(\overline{N}_I - \overline{N}_E)k$. (iv) The sign of the derivative of $\Delta(\overline{N}_I - m, \overline{N}_E - m)$ with respect to δ depends on the sign of $(1 - m(1-\delta)\delta^m - \delta^m)(\theta_I - \theta_E) = (1-\delta)^2(\theta_I - \theta_E)\sum_{j=1}^{m}(j\delta^{j-1})$.

7.4 Proof of Proposition 5

For part (i) of the proposition, recall that $\Delta(N_I, N_E)$ is proportional to θ and positive when $N_I > N_E$. For part (ii), suppose $N_I + k \leq \underline{N}$. Given $N_I > N_E$, this implies $N_E + k < \underline{N}$, so both firms will remain on the flat part of the learning curve after the change. Then $E(N_I + k) = \overline{N} - \underline{N}$ and $E(N_E + k) = \overline{N} - \underline{N}$, and so

$$\Delta(N_I + k, N_E + k) - \Delta(N_I, N_E) = \frac{(1-\delta^k)(1-\delta^{\overline{N}-\underline{N}})(\delta^{\overline{N}-N_I} - \delta^{\overline{N}-N_E})\theta}{(1-\delta)\delta^{\overline{N}-\underline{N}+k-1}} > 0.$$

For part (iii) suppose instead $N_I > \underline{N}$ and $N_E > \underline{N}$, so both firms start on the increasing part of the learning curve before the change. Then $E(N_I) = \overline{N} - N_I$ and $E(N_E) = \overline{N} - N_E$, and so

$$\Delta(N_I + k, N_E + k) - \Delta(N_I, N_E) = -\frac{(1-\delta^k)(\delta^{\overline{N}-N_I+1} - \delta^{\overline{N}-N_E+1})\theta}{(1-\delta)\delta^k},$$

which is negative given $N_I > N_E$.

Finally, for part (iv) we have

$$\Delta(N_I, N_E) = \left(N_I - N_E + \frac{\delta^{\overline{N}-N_E+1} - \delta^{\overline{N}-N_I+1}}{1-\delta}\right)\theta = \left(N_I - N_E - \sum_{j=\overline{N}-N_I+1}^{\overline{N}-N_E}\delta^j\right)\theta,$$

which is clearly decreasing in δ.

7.5 Proof of Proposition 6

As discussed in the text, the analysis can be conducted with a representative consumer.

First, we prove the result for $(N_I, N_E) = (\overline{N}_I, \overline{N}_E)$, i.e. when both firms have reached their respective threshold with respect to a particular consumer. Suppose I charges p^I and E charges p^E in the current period. Then the consumer chooses I if and only if

$$s_I + f_I(\overline{N}_I) - p^I + \delta u(\overline{N}_I, \overline{N}_E) > s_E + f_E(\overline{N}_E) - p^E + \delta u(\overline{N}_I, \overline{N}_E),$$

where $u(N_I, N_E)$ is the value function for the consumer, i.e. the PDV of surplus a consumer expects to obtain from making her optimal choices in every stage when firms play their equilibrium strategies and the consumer has previously bought N_I times from firm I and N_E times from firm E. Here $u(\overline{N}_I, \overline{N}_E)$ appears on both sides of the consumer's choice problem since the future surplus the consumer can obtain following either choice in the current period is the same given that both firms have already reached their learning threshold. Furthermore, since both firms are at their respective learning threshold, no matter who wins, the state will be the same next period. This implies neither firm is willing to price below cost in the current period. Thus, given Bertrand competition and the fact $u(\overline{N}_I, \overline{N}_E)$ cancels out in the consumer's choice problem, we obtain

$$\begin{aligned} V^I(\overline{N}_I, \overline{N}_E) &= \delta V^I(\overline{N}_I, \overline{N}_E) + \max\left\{s_I + f_I(\overline{N}_I) - s_E - f_E(\overline{N}_E), 0\right\} \\ V^E(\overline{N}_I, \overline{N}_E) &= \delta V^E(\overline{N}_I, \overline{N}_E) + \max\left\{s_E + f_E(\overline{N}_E) - s_I - f_I(\overline{N}_I), 0\right\}. \end{aligned}$$

Clearly, the solution is

$$\begin{aligned} V^I(\overline{N}_I, \overline{N}_E) &= \max\left\{\frac{s_I - s_E + f_I(\overline{N}_I) - f_E(\overline{N}_E)}{1-\delta}, 0\right\} \\ V^E(\overline{N}_I, \overline{N}_E) &= \max\left\{\frac{s_E - s_I + f_E(\overline{N}_E) - f_I(\overline{N}_I)}{1-\delta}, 0\right\} \end{aligned}$$

and we have

$$\Delta(\overline{N}_I, \overline{N}_E) = f_I(\overline{N}_I) - f_E(\overline{N}_E).$$

Note if $s_I - s_E + f_I(\overline{N}_I) - f_E(\overline{N}_E) > 0$, then I wins in every period, and it leaves consumers with the surplus that they could get buying from E, which is $s_E + f_E(\overline{N}_E) - c$ in every period since $V^E(\overline{N}_I, \overline{N}_E) = 0$ in this case. On the other hand, if $s_E - s_I + f_E(\overline{N}_E) - f_I(\overline{N}_I) \geq 0$, then E wins every period, and it leaves consumers with the surplus that they could get buying from I, which is $s_I + f_I(\overline{N}_I) - c$ in every period since $V^I(\overline{N}_I, \overline{N}_E) = 0$ in this case. This implies

$$u(\overline{N}_I, \overline{N}_E) = \frac{\min\left\{s_I + f_I(\overline{N}_I), s_E + f_E(\overline{N}_E)\right\} - c}{1-\delta}.$$

Suppose now that the result in the Proposition holds for the state $(\overline{N}_I, N_E)$ with $1 \leq N_E \leq \overline{N}_E$. Specifically, this means

$$\Delta(\overline{N}_I, N_E) = f_I(\overline{N}_I) - (1-\delta)\left(\sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E + j)\right) - \delta^{\overline{N}_E - N_E + 1} f_E(\overline{N}_E)$$

$$V^I(\overline{N}_I, N_E) = \frac{1}{1-\delta}\max\left\{s_I - s_E + \Delta(\overline{N}_I, N_E), 0\right\}$$

$$V^E(\overline{N}_I, N_E) = \frac{1}{1-\delta}\max\left\{s_E - s_I - \Delta(\overline{N}_I, N_E), 0\right\}$$

$$u(\overline{N}_I, N_E) = \min \left\{ f_I(\overline{N}_I) + \frac{s_I + \delta f_I(\overline{N}_I) - c}{1-\delta}, \sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E + j) + \frac{s_E + \delta^{\overline{N}_E - N_E + 1} f_E(\overline{N}_E) - c}{1-\delta} \right\}.$$

Consider now the state $(\overline{N}_I, N_E - 1)$. The consumer chooses I in the current period if and only if

$$s_I + f_I(\overline{N}_I) - p^I + \delta u(\overline{N}_I, N_E - 1) > s_E + f_E(N_E - 1) - p^E + \delta u(\overline{N}_I, N_E).$$

I is willing to set p^I down to the point where it is indifferent between winning or losing the current period, i.e. $p^I = c - \delta(V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E))$. Similarly, E is willing to set p^E down to $p^E = c - \delta(V^E(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1))$. Thus, I wins the current period if and only if

$$\begin{aligned} & s_I + f_I(\overline{N}_I) + \delta(V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E)) + \delta u(\overline{N}_I, N_E - 1) \\ > \quad & s_E + f_E(N_E - 1) + \delta(V^E(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1)) + \delta u(\overline{N}_I, N_E). \end{aligned} \tag{12}$$

If (12) holds, then E sets $p^E = c - \delta(V^E(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1))$ and I sets

$$p^I = c + s_I + f_I(\overline{N}_I) - s_E - f_E(N_E - 1) + \delta(u(\overline{N}_I, N_E - 1) + V^E(\overline{N}_I, N_E - 1) - u(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E)),$$

which implies

$$u(\overline{N}_I, N_E - 1) = s_E + f_E(N_E - 1) - c + \delta(V^E(\overline{N}_I, N_E) + u(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1)).$$

If (12) does not hold, then I sets $p^I = c - \delta(V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E))$ and E sets

$$p^E = c + s_E + f_E(N_E - 1) - s_I - f_I(\overline{N}_I) + \delta(u(\overline{N}_I, N_E) + V^I(\overline{N}_I, N_E) - u(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E - 1)),$$

which implies

$$u(\overline{N}_I, N_E - 1) = s_I + f_I(\overline{N}_I) - c + \delta(u(\overline{N}_I, N_E - 1) + V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E)).$$

The bottomline is that

$$u(\overline{N}_I, N_E - 1) = \min \left\{ \begin{array}{c} s_I + f_I(\overline{N}_I) + \delta(V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E) + u(\overline{N}_I, N_E - 1)), \\ s_E + f_E(N_E - 1) + \delta(V^E(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1) + u(\overline{N}_I, N_E)) \end{array} \right\} - c$$

$$V^I(\overline{N}_I, N_E - 1) = \delta V^I(\overline{N}_I, N_E) + \max \left\{ \begin{array}{c} s_I + f_I(\overline{N}_I) - s_E - f_E(N_E - 1) \\ +\delta \left(\begin{array}{c} V^I(\overline{N}_I, N_E - 1) + V^E(\overline{N}_I, N_E - 1) + u(\overline{N}_I, N_E - 1) \\ -V^I(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E) - u(\overline{N}_I, N_E) \end{array} \right), 0 \end{array} \right\}$$

$$V^E(\overline{N}_I, N_E-1) = \delta V^E(\overline{N}_I, N_E-1) + \max\left\{ \begin{array}{c} s_E + f_E(N_E-1) - s_I - f_I(\overline{N}_I) \\ +\delta \left(\begin{array}{c} V^I(\overline{N}_I, N_E) + V^E(\overline{N}_I, N_E) + u(\overline{N}_I, N_E) \\ -V^I(\overline{N}_I, N_E-1) - V^E(\overline{N}_I, N_E-1) - u(\overline{N}_I, N_E-1) \end{array} \right) \end{array}, 0 \right\}$$

There are two cases: $V^E(\overline{N}_I, N_E-1) = 0$ and $V^E(\overline{N}_I, N_E-1) > 0$.

Suppose $V^E(\overline{N}_I, N_E-1) = 0$, which means I wins the current period, so

$$u(\overline{N}_I, N_E-1) = s_E + f_E(N_E-1) + \delta(V^E(\overline{N}_I, N_E) + u(\overline{N}_I, N_E)) - c$$

$$V^I(\overline{N}_I, N_E-1) = s_I + f_I(\overline{N}_I) - s_E - f_E(N_E-1) + \delta \left(\begin{array}{c} V^I(\overline{N}_I, N_E-1) + u(\overline{N}_I, N_E-1) \\ -V^E(\overline{N}_I, N_E) - u(\overline{N}_I, N_E) \end{array} \right)$$

Plugging the expression of $u(\overline{N}_I, N_E-1)$ into that of $V^I(\overline{N}_I, N_E-1)$, we obtain

$$V^I(\overline{N}_I, N_E-1) = \frac{s_I + f_I(\overline{N}_I) - (1-\delta)(s_E + f_E(N_E-1) + \delta(V^E(\overline{N}_I, N_E) + u(\overline{N}_I, N_E))) - \delta c}{1-\delta}.$$

Using

$$V^E(\overline{N}_I, N_E) + u(\overline{N}_I, N_E) = \sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E+j) + \frac{s_E + \delta^{\overline{N}_E - N_E+1} f_E(\overline{N}_E) - c}{1-\delta},$$

we obtain

$$V^I(\overline{N}_I, N_E-1) = \frac{s_I - s_E + \Delta(\overline{N}_I, N_E-1)}{1-\delta}.$$

$$u(\overline{N}_I, N_E-1) = \frac{s_E + \delta^{\overline{N}_E - N_E+2} f_E(\overline{N}_E) - c}{1-\delta} + \sum_{j=0}^{\overline{N}_E - N_E+1} \delta^j f_E(N_E-1+j).$$

Now suppose $V^E(\overline{N}_I, N_E-1) > 0$, which means E wins the current period, so $V^I(\overline{N}_I, N_E-1) = \delta V^I(\overline{N}_I, N_E)$, which implies $V^I(\overline{N}_I, N_E-1) = V^I(\overline{N}_I, N_E) = 0$ because we must have $V^I(\overline{N}_I, N_E-1) \geq V^I(\overline{N}_I, N_E)$. Furthermore, the expressions of $u(\overline{N}_I, N_E-1)$ and $V^E(\overline{N}_I, N_E-1)$ become

$$u(\overline{N}_I, N_E-1) = \frac{s_I + f_I(\overline{N}_I) - c}{1-\delta}$$

$$\begin{aligned} V^E(\overline{N}_I, N_E-1) &= s_E + f_E(N_E-1) - s_I - f_I(\overline{N}_I) + \delta(V^E(\overline{N}_I, N_E) + u(\overline{N}_I, N_E) - u(\overline{N}_I, N_E-1)) \\ &= \frac{s_E - s_I - \Delta(\overline{N}_I, N_E-1)}{1-\delta}. \end{aligned}$$

Combining the two cases $V^E(\overline{N}_I, N_E - 1) = 0$ and $V^E(\overline{N}_I, N_E - 1) > 0$, we obtain

$$\begin{aligned} V^I(\overline{N}_I, N_E - 1) &= \frac{1}{1-\delta}\max\left\{s_I - s_E + \Delta(\overline{N}_I, N_E - 1), 0\right\} \\ V^E(\overline{N}_I, N_E - 1) &= \frac{1}{1-\delta}\max\left\{s_E - s_I - \Delta(\overline{N}_I, N_E - 1), 0\right\} \end{aligned}$$

$$u(\overline{N}_I, N_E - 1) = \min\left\{\frac{s_I + f_I(\overline{N}_I) - c}{1-\delta}, \frac{s_E + \delta^{\overline{N}_E - N_E + 2} f_E(\overline{N}_E) - c}{1-\delta} + \sum_{j=0}^{\overline{N}_E - N_E + 1} \delta^j f_E(N_E - 1 + j)\right\}.$$

Thus, by induction, we can conclude that the result holds for all states $(\overline{N}_I, N_E)$ with $0 \le N_E \le \overline{N}_E$. By symmetry, it also holds for all states $(N_I, \overline{N}_E)$ such that $0 \le N_I \le \overline{N}_I$.

Now consider the state (N_I, N_E) with $0 \le N_I \le \overline{N}_I - 1$ and $0 \le N_E \le \overline{N}_E - 1$, and suppose the result in the Proposition holds for the states $(N_I + 1, N_E)$ and $(N_I, N_E + 1)$. Using the same reasoning as above, we have

$$u(N_I, N_E) = \min\left\{\begin{array}{c} s_I + f_I(N_I) + \delta(V^I(N_I + 1, N_E) - V^I(N_I, N_E + 1) + u(N_I + 1, N_E)), \\ s_E + f_E(N_E) + \delta(V^E(N_I, N_E + 1) - V^E(N_I + 1, N_E) + u(N_I, N_E + 1)) \end{array}\right\} - c$$

$$V^I(N_I, N_E) = \delta V^I(N_I, N_E + 1) + \max\left\{\begin{array}{c} s_I + f_I(N_I) - s_E - f_E(N_E) \\ +\delta\left(\begin{array}{c} V^I(N_I + 1, N_E) + V^E(N_I + 1, N_E) + u(N_I + 1, N_E) \\ -V^I(N_I, N_E + 1) - V^E(N_I, N_E + 1) - u(N_I, N_E + 1) \end{array}\right), 0 \end{array}\right\}$$

$$V^E(N_I, N_E) = \delta V^E(N_I + 1, N_E) + \max\left\{\begin{array}{c} s_I + f_I(N_I) - s_E - f_E(N_E) \\ +\delta\left(\begin{array}{c} V^E(N_I, N_E + 1) + V^I(N_I, N_E + 1) + u(N_I, N_E + 1) \\ -V^E(N_I + 1, N_E) - V^I(N_I + 1, N_E) - u(N_I + 1, N_E) \end{array}\right), 0 \end{array}\right\}.$$

There are two cases. Case (i). Suppose first I wins the current period, which means $V^E(N_I, N_E) = \delta V^E(N_I + 1, N_E)$, so $V^E(N_I, N_E) = V^E(N_I + 1, N_E) = 0$ because we must have $V^E(N_I, N_E) \ge V^E(N_I + 1, N_E)$. Thus, we have

$$u(N_I, N_E) = s_E + f_E(N_E) + \delta(V^E(N_I, N_E + 1) + u(N_I, N_E + 1)) - c$$

$$V^I(N_I, N_E) = s_I + f_I(N_I) - s_E - f_E(N_E) + \delta\left(\begin{array}{c} V^I(N_I + 1, N_E) + u(N_I + 1, N_E) \\ -V^E(N_I, N_E + 1) - u(N_I, N_E + 1) \end{array}\right).$$

Using the induction hypothesis, we have

$$V^I(N_I + 1, N_E) + u(N_I + 1, N_E) = \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + 1 + j) + \frac{s_I + \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - c}{1-\delta}$$

$$V^E(N_I, N_E + 1) + u(N_I, N_E + 1) = \sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + 1 + j) + \frac{s_E + \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E) - c}{1-\delta}.$$

So we obtain

$$u(N_I, N_E) = \sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E + j) + \frac{s_E + \delta^{\overline{N}_E - N_E + 1} f_E(\overline{N}_E) - c}{1 - \delta}$$

$$\begin{aligned} V^I(N_I, N_E) &= \frac{s_I + \delta^{\overline{N}_I - N_I + 1} f_I(\overline{N}_I) - s_E - \delta^{\overline{N}_E - N_E + 1} f_E(\overline{N}_E)}{1 - \delta} \\ &\quad + \sum_{j=0}^{\overline{N}_I - N_I} \delta^j f_I(N_I + j) - \sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E + j) \\ &= \frac{s_I - s_E + \Delta(N_I, N_E)}{1 - \delta}. \end{aligned}$$

Case (ii). Suppose instead E wins the current period. By symmetry, we have $V^I(N_I, N_E) = V^I(N_I, N_E + 1) = 0$ and

$$\begin{aligned} u(N_I, N_E) &= \sum_{j=0}^{\overline{N}_I - N_I} \delta^j f_I(N_I + j) + \frac{s_I + \delta^{\overline{N}_I - N_I + 1} f_I(\overline{N}_I) - c}{1 - \delta} \\ V^E(N_I, N_E) &= \frac{s_E - s_I - \Delta(N_I, N_E)}{1 - \delta}. \end{aligned}$$

Combining (i) and (ii), we obtain

$$\begin{aligned} V^I(N_I, N_E) &= \max\left\{\frac{s_I - s_E + \Delta(N_I, N_E)}{1 - \delta}, 0\right\} \\ V^E(N_I, N_E) &= \max\left\{\frac{s_E - s_I - \Delta(N_I, N_E)}{1 - \delta}, 0\right\} \end{aligned}$$

$$u(N_I, N_E) = \min\left\{ \begin{array}{c} \sum_{j=0}^{\overline{N}_I - N_I} \delta^j f_I(N_I + j) + \frac{s_I + \delta^{\overline{N}_I - N_I + 1} f_I(\overline{N}_I) - c}{1 - \delta}, \\ \sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E + j) + \frac{s_E + \delta^{\overline{N}_E - N_E + 1} f_E(\overline{N}_E) - c}{1 - \delta} \end{array} \right\}.$$

By repeated application of the induction hypothesis, the result holds for all states (N_I, N_E), with $0 \leq N_I \leq \overline{N}_I$ and $0 \leq N_E \leq \overline{N}_E$.

Finally, we can confirm that consumers do not want to switch firms along the equilibrium path. Suppose the current state is (N_I, N_E). If $s_E - s_I < \Delta(N_I, N_E)$, then consumers choose I in the current period, so next period the state will be $(N_I + 1, N_E)$. Since $\Delta(N_I, N_E)$ is increasing in N_I, this means in the subsequent period consumers will choose I again, and so on until $N_I = \overline{N}_I$. When the state is $(\overline{N}_I, N_E)$, all subsequent periods will have an identical state and outcome, in which consumers choose I. Conversely, if $s_E - s_I \geq \Delta(N_I, N_E)$, then consumers choose E in the current period, so next period the state will be $(N_I, N_E + 1)$. Since $\Delta(N_I, N_E)$ is decreasing in N_E, this means in the subsequent period consumers will choose E again, and so on until $N_E = \overline{N}_E$.

At the state $(N_I, \overline{N}_E)$, all subsequent periods will have an identical state and outcome, in which consumers choose E.

7.6 Proof of Proposition 7

There are four different cases, depending on the value of $s_E - s_I$. If $s_E - s_I < \Delta(N_I, N_E + 1)$ then $V^I(N_I, N_E)$ is higher under across-user learning since

$$\frac{s_I - s_E}{1-\delta} + \frac{\Delta(N_I, N_E) - \delta\Delta(N_I, N_E + 1)}{(1-\delta)^2} > \frac{s_I - s_E + \Delta(N_I, N_E)}{1-\delta} > 0$$

given $\Delta(N_I, N_E) > \Delta(N_I, N_E + 1)$. If $\Delta(N_I, N_E + 1) \leq s_E - s_I < \Delta(N_I, N_E)$, then $V^I(N_I, N_E)$ is higher under across-user learning since

$$\frac{s_I - s_E + \Delta(N_I, N_E)}{(1-\delta)^2} > \frac{s_I - s_E + \Delta(N_I, N_E)}{1-\delta} > 0.$$

If $\Delta(N_I, N_E) \leq s_E - s_I < \Delta(N_I + 1, N_E)$, then $V^E(N_I, N_E)$ is higher under across-user learning since

$$\frac{s_E - s_I - \Delta(N_I, N_E)}{(1-\delta)^2} > \frac{s_E - s_I - \Delta(N_I, N_E)}{1-\delta} > 0.$$

Finally, if $s_E - s_I \geq \Delta(N_I + 1, N_E)$, then $V^E(N_I, N_E)$ is higher under across-user learning since

$$\frac{s_E - s_I}{1-\delta} - \frac{\Delta(N_I, N_E) - \delta\Delta(N_I + 1, N_E)}{(1-\delta)^2} > \frac{s_E - s_I - \Delta(N_I, N_E)}{1-\delta} > 0$$

given $\Delta(N_I + 1, N_E) > \Delta(N_I, N_E)$.

7.7 Proof of Proposition 11

The case with $N_I = N_E = n_I = n_E = 1$ was dealt with in the text. Consider the case with $N_I = n_I = 1$ and $N_E = n_E = 0$. Given the learning functions are both strictly increasing, the only two possible (pure strategy) equilibria are as follows:

- All consumers choose I in every period. This is an equilibrium if and only if no individual consumer can do better deviating to E, which if it were better would lead to that consumer choosing E in all subsequent periods. Since the payoffs on the equilibrium path and on the deviation path are the same every periods, this deviation can be ruled out if and only if

$$s_I + f^A(1) + f^W(1) > s_E + \delta f^W(1).$$

- All consumers choose E in every period. This is an equilibrium if and only if no individual consumer can do better deviating to I for at least one period. The gain from deviating to I is greatest in the first period since after the first period the consumer can get the across-user learning by choosing E. Thus, if a deviation to I for one period followed by choosing E in all subsequent periods is not better for a consumer than neither will any other deviation that involves choosing I for more than one period. Thus, to rule out a deviation we just need that

$$\frac{s_E + \delta\left(f^A(1) + f^W(1)\right)}{1-\delta} \geq s_I + f^A(1) + f^W(1) + \delta\frac{s_E + f^A(1) + \delta f^W(1)}{1-\delta},$$

or equivalently that

$$s_E + \delta f_E^W(1) \geq s_I + f_I^A(1) + f_I^W(1).$$

Thus, the unique equilibrium is for all consumers to choose E in all periods if $s_E + \delta f_E^W(1) \geq s_I + f_I^A(1) + f_I^W(1)$, and otherwise choose I in all periods.

Now, consider the case with $N_I = N_E = n_I = n_E = 0$. Again, there are only two possible equilibria[8]:

- All consumers choose I in every period. This is an equilibrium if and only if no individual consumer can do better deviating to choose E for at least one period. If a consumer is to deviate to E it does best to do so in period 1, since after period 1 the surplus from choosing I will be greater. There are two possibilities. A consumer may deviate to E for the first period to take advantage of the possibility $s_E > s_I$, but then switch to I in subsequent periods to take advantage of across-user learning, or it may deviate to E for all periods. To rule out these two deviations requires

$$s_E - s_I < \delta \min\left\{f^A(1), f^W(1)\right\}.$$

- All consumers choose E in all periods. By symmetry, this is an equilibrium if and only if

$$s_E - s_I \geq -\delta \min\left\{f^A(1), f^W(1)\right\}.$$

In summary:

- if $s_E - s_I < -\delta \min\left\{f^A(1), f^W(1)\right\}$, then the unique equilibrium is for all consumers to choose I in all periods.
- if $s_E - s_I \geq \delta \min\left\{f^A(1), f^W(1)\right\}$, then the unique equilibrium is for all consumers to choose E in all periods.
- if $-\delta \min\left\{f^A(1), f^W(1)\right\} \leq s_E - s_I < \delta \min\left\{f^A(1), f^W(1)\right\}$, then both equilibria exist.

[8]It is straightforward to obtain a contradiction from any equilibrium in which consumers choose one firm in period 1 and then choose a different firm in a later period. We show this property more generally in the Online Appendix.

To illustrate that expectations matter, suppose first consumers hold Pareto beliefs. This implies whenever there are multiple equilibria, consumers coordinate on the equilibrium that yields higher surplus for them. Given learning functions are identical, choosing E for all periods over I yields higher consumer surplus whenever $s_E - s_I \geq 0$. Thus, with Pareto beliefs, E wins if and only if $s_E - s_I \geq 0$. Contrast this with beliefs favoring I. Then I wins whenever it is an equilibrium for I to win, so E wins if and only if $s_E - s_I \geq \delta \min \left\{ f^A(1), f^W(1) \right\}$.

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