Online Appendix

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We provide detailed proofs of material that is not included in the main text of our paper "Data-enabled learning, network effects and competitive advantage".

A Finite number of periods with pure across-user learning

In this section, we derive the threshold for the game with pure across-user learning and a finite number of periods. Denote the two firms' profits when there are T periods remaining and the current state is (N_I, N_E) by $\Pi^I(N_I, N_E, T)$ and $\Pi^E(N_I, N_E, T)$.

For $T = 1$ we have

$$\begin{aligned} \Pi^I(N_I, N_E, 1) &= \max\{s_I + f_I(N_I) - s_E - f_E(N_E), 0\} \\ \Pi^E(N_I, N_E, 1) &= \max\{s_E + f_E(N_E) - s_I - f_I(N_I), 0\}, \end{aligned}$$

so

$$\Delta(N_I, N_E, 1) = f_I(N_I) - f_E(N_E).$$

Suppose now that for some $T > 0$ and any state (N_I, N_E) we have

$$\Delta(N_I, N_E, T) = (1-\delta)\left(\sum_{j=0}^{T-1} \frac{\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T + T\delta^{T+1}}\left(f_I(N_I+j) - f_E(N_E+j)\right)\right)$$

$$\Pi^I(N_I, N_E, T) = \begin{cases} \sum_{j=0}^{T-1} \delta^j\left(s_I + f_I(N_I+j) - s_E - f_E(N_E)\right) & \text{if} \quad s_E - s_I < \Delta(N_I, N_E+1, T-1) \\ \left(1+2\delta+...+T\delta^{T-1}\right)\left(s_I - s_E + \Delta(N_I, N_E, T)\right) & \text{if} \quad \begin{array}{c}\Delta(N_I, N_E+1, T-1) \le s_E - s_I \\ < \Delta(N_I, N_E, T)\end{array} \\ 0 & \text{if} \quad s_E - s_I \ge \Delta(N_I, N_E, T) \end{cases}$$

$$\Pi^E(N_I, N_E, T) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta(N_I, N_E, T) \\ \left(1+2\delta+...+T\delta^{T-1}\right)\left(s_E - s_I - \Delta(N_I, N_E, T)\right) & \text{if} \quad \begin{array}{c}\Delta(N_I, N_E, T) \le s_E - s_I \\ < \Delta(N_I+1, N_E, T-1)\end{array} \\ \sum_{j=0}^{T-1} \delta^j\left(s_E + f_E(N_E+j) - s_I - f_I(N_I)\right) & \text{if} \quad s_E - s_I \ge \Delta(N_I+1, N_E, T-1) \end{cases}.$$

At the end of this section we show that $\Delta(N_I, N_E+1, T-1) < \Delta(N_I, N_E, T) < \Delta(N_I+1, N_E, T-1)$ for any (N_I, N_E) and $T \ge 1$.

Consider the game with $T+1$ periods starting from state (N_I, N_E). I wins iff

$$\begin{aligned} & s_I + f_I(N_I) + \delta\left(\Pi^I(N_I+1, N_E, T) - \Pi^I(N_I, N_E+1, T)\right) \\ > \; & s_E + f_E(N_E) + \delta\left(\Pi^E(N_I, N_E+1, T) - \Pi^E(N_I+1, N_E, T)\right), \end{aligned}$$

so

$$\begin{aligned} \Pi^I(N_I, N_E, T+1) &= \delta\Pi^I(N_I, N_E+1, T) + \max\left\{ \begin{array}{c} s_I - s_E + f_I(N_I) - f_E(N_E) \\ +\delta\left(\begin{array}{c} \Pi^I(N_I+1, N_E, T) + \Pi^E(N_I+1, N_E, T) \\ -\Pi^I(N_I, N_E+1, T) - \Pi^E(N_I, N_E+1, T) \end{array}\right), 0 \end{array}\right\} \\ \Pi^E(N_I, N_E, T+1) &= \delta\Pi^E(N_I+1, N_E, T) + \max\left\{ \begin{array}{c} s_E - s_I + f_E(N_E) - f_I(N_I) \\ +\delta\left(\begin{array}{c} \Pi^I(N_I, N_E+1, T) + \Pi^E(N_I, N_E+1, T) \\ -\Pi^I(N_I+1, N_E, 1) - \Pi^E(N_I+1, N_E, T) \end{array}\right), 0 \end{array}\right\}. \end{aligned}$$

Using the induction result for the game with T periods to write the expressions of $\Pi^I(N_I+1, N_E, T)$ and $\Pi^E(N_I, N_E+1, T)$, straightforward calculations yield

$$\Pi^I(N_I, N_E, T+1) = \begin{cases} \sum_{j=0}^{T} \delta^j \left(s_I + f_I(N_I+j) - s_E - f_E(N_E)\right) & \text{if} \quad s_E - s_I < \Delta(N_I, N_E+1, T) \\ \left(1 + 2\delta + ... + (T+1)\delta^T\right)\left(\begin{array}{c} s_I - s_E \\ +\Delta(N_I, N_E, T+1) \end{array}\right) & \text{if} \quad \begin{array}{c} \Delta(N_I, N_E+1, T) \leq s_E - s_I \\ < \Delta(N_I, N_E, T+1) \end{array} \\ 0 & \text{if} \quad s_E - s_I \geq \Delta(N_I, N_E, T+1) \end{cases}$$

$$\Pi^E(N_I, N_E, T+1) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta(N_I, N_E, T+1) \\ \left(1 + 2\delta + ... + (T+1)\delta^T\right)\left(\begin{array}{c} s_E - s_I \\ -\Delta(N_I, N_E, T+1) \end{array}\right) & \text{if} \quad \begin{array}{c} \Delta(N_I, N_E, T+1) \leq s_E - s_I \\ < \Delta(N_I+1, N_E, T) \end{array} \\ \sum_{j=0}^{T} \delta^j \left(s_E + f_E(N_E+j) - s_I - f_I(N_I)\right) & \text{if} \quad s_E - s_I \geq \Delta(N_I+1, N_E, T) \end{cases}$$

so

$$\Delta(N_I, N_E, T+1) = (1-\delta)\left(\sum_{j=0}^{T} \frac{\delta^j\left(1 - \delta^{T+1-j}\right)}{1 - (T+2)\delta^{T+1} + (T+1)\delta^{T+2}} \left(f_I(N_I+j) - f_E(N_E+j)\right)\right)$$

Thus, by induction, the expressions above hold for any $T \geq 1$.

The last thing remaining is to confirm that $\Delta(N_I, N_E+1, T-1) < \Delta(N_I, N_E, T) < \Delta(N_I+1, N_E, T-1)$

for any (N_I, N_E) and $T \geq 1$. To do so, write

$$\begin{aligned}\Delta(N_I, N_E, T) - \Delta(N_I, N_E+1, T-1) &= \sum_{j=0}^{T-1} \frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T + T\delta^{T+1}} \left(f_I(N_I+j) - f_E(N_E+j)\right) \\ &\quad - \sum_{j=0}^{T-2} \frac{(1-\delta)\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T} \left(f_I(N_I+j) - f_E(N_E+1+j)\right)\end{aligned}$$

$$\begin{aligned}&= \sum_{j=0}^{T-1} \frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T + T\delta^{T+1}} f_I(N_I+j) - \sum_{j=0}^{T-1} \frac{(1-\delta)\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T} f_I(N_I+j) \\ &\quad + \sum_{j=0}^{T-2} \frac{(1-\delta)\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T} f_E(N_E+1+j) - \sum_{j=0}^{T-1} \frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T + T\delta^{T+1}} f_E(N_E+j)\end{aligned}$$

It is easily verified that this expression is positive for $T = 1$ and $T = 2$. So assume $T \geq 3$. Recall that

$$\sum_{j=0}^{T-1} \frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T + T\delta^{T+1}} = \sum_{j=0}^{T-1} \frac{(1-\delta)\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T} = \sum_{j=0}^{T-2} \frac{(1-\delta)\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T} = 1.$$

Furthermore, it is straightforward to verify that there exists $j^* \in [1, ..., T-2]$ such that

$$\frac{(1-\delta)\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T} > \frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T + T\delta^{T+1}}$$

for all $0 \leq j \leq j^*$ and

$$\frac{(1-\delta)\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T} \leq \frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T + T\delta^{T+1}}$$

for all $j^* < j \leq T-1$.

Thus, we have

$$\begin{aligned}&\sum_{j=0}^{T-1} \frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T + T\delta^{T+1}} f_I(N_I+j) - \sum_{j=0}^{T-1} \frac{(1-\delta)\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T} f_I(N_I+j) \\ &= \sum_{j=0}^{j^*} \left(\frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T + T\delta^{T+1}} - \frac{(1-\delta)\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T} \right) f_I(N_I+j) \\ &\quad + \sum_{j=j^*+1}^{T-1} \left(\frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T + T\delta^{T+1}} - \frac{(1-\delta)\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T} \right) f_I(N_I+j)\end{aligned}$$

$$
\begin{aligned}
> & \sum_{j=0}^{j^*}\left(\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}-\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T}\right)f_I\left(N_I+j^*\right) \\
& +\sum_{j=j^*+1}^{T-1}\left(\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}-\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T}\right)f_I\left(N_I+j^*\right) \\
= & \ 0.
\end{aligned}
$$

And

$$
\begin{aligned}
& \sum_{j=0}^{T-2}\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T}f_E\left(N_E+1+j\right)-\sum_{j=0}^{T-1}\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}f_E\left(N_E+j\right) \\
= & \sum_{j=1}^{T-1}\frac{(1-\delta)\,\delta^{j-1}\left(1-\delta^{T-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T}f_E\left(N_E+j\right)-\sum_{j=0}^{T-1}\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}f_E\left(N_E+j\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{j=1}^{T-1}\left(\frac{(1-\delta)\,\delta^{j-1}\left(1-\delta^{T-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T}-\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}\right)f_E\left(N_E+j\right)-\frac{(1-\delta)\left(1-\delta^T\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}f_E\left(N_E\right) \\
> & \sum_{j=1}^{T-1}\left(\frac{(1-\delta)\,\delta^{j-1}\left(1-\delta^{T-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T}-\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}\right)f_E\left(N_E\right)-\frac{(1-\delta)\left(1-\delta^T\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}f_E\left(N_E\right) \\
= & \ 0,
\end{aligned}
$$

where the inequality follows from the observation that $\frac{(1-\delta)\delta^{j-1}\left(1-\delta^{T-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T}>\frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T+T\delta^{T+1}}$ for all j.

We can therefore conclude that

$$
\begin{aligned}
& \sum_{j=0}^{T-1}\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}f_I\left(N_I+j\right)-\sum_{j=0}^{T-1}\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T}f_I\left(N_I+j\right) \\
& +\sum_{j=0}^{T-2}\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T}f_E\left(N_E+1+j\right)-\sum_{j=0}^{T-1}\frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}f_E\left(N_E+j\right) \\
> & \ 0,
\end{aligned}
$$

so $\Delta\left(N_I,N_E,T\right)>\Delta\left(N_I,N_E+1,T-1\right)$.

The symmetry of I and E then implies that $\Delta\left(N_I,N_E,T\right)<\Delta\left(N_I+1,N_E,T-1\right)$.

B Finite number of periods with pure within-user learning

In this section, we prove that the outcome of the game with pure within-user learning and a finite time horizon is socially optimal (unlike the case with across-user learning and a finite time horizon).

Recall that the state (N_I, N_E) here means that the representative consumer has previously purchased N_i times from firm i, for $i = I, E$. If there are t periods remaining and the state is (N_I, N_E), the two firms' profits are denoted $\Pi^I(N_I, N_E, t)$ and $\Pi^E(N_I, N_E, t)$, while the PDV of the net surplus derived by the representative consumer is denoted $u(N_I, N_E, t)$.

Consider first $t = 1$. We have

$$\begin{aligned} \Pi^I(N_I, N_E, 1) &= \max\{s_I + f_I(N_I) - s_E - f_E(N_E), 0\} \\ \Pi^E(N_I, N_E, 1) &= \max\{s_E + f_E(N_E) - s_I - f_I(N_I), 0\}. \end{aligned}$$

So

$$\begin{aligned} \Delta(N_I, N_E, 1) &= f_I(N_I) - f_E(N_E) \\ u(N_I, N_E, 1) &= \min\{s_I - c + f_I(N_I), s_E - c + f_E(N_E)\}. \end{aligned}$$

Thus, the result holds for $t = 1$.

Now suppose for some $t > 0$, we have

$$\begin{aligned} \Pi^I(N_I, N_E, t) &= \max\left\{\sum_{j=0}^{t-1} \delta^j \left(s_I + f_I(N_I + j) - s_E - f_E(N_E + j)\right), 0\right\} \\ \Pi^E(N_I, N_E, t) &= \max\left\{\sum_{j=0}^{t-1} \delta^j \left(s_E + f_E(N_E + j) - s_I - f_I(N_I + j)\right), 0\right\} \end{aligned}$$

$$\begin{aligned} \Delta(N_I, N_E, t) &= \frac{\sum_{j=0}^{t-1} \delta^j \left(f_I(N_I + j) - f_E(N_E + j)\right)}{\sum_{j=0}^{t-1} \delta^j} \\ u(N_I, N_E, t) &= \min\left\{\sum_{j=0}^{t-1} \delta^j \left(s_I - c + f_I(N_I + j)\right), \sum_{j=0}^{t-1} \delta^j \left(s_E - c + f_E(N_E + j)\right)\right\} \end{aligned}$$

for all (N_I, N_E). Note that $\Delta(N_I, N_E, t)$ defined above is the socially optimum cutoff for the game starting in state (N_I, N_E) and with t periods left.

Consider now the game starting with state (N_I, N_E) and having $t+1$ periods left. I wins iff

$$\begin{aligned}
& s_I + f_I(N_I) + \delta\left(\Pi^I(N_I+1, N_E, t) - \Pi^I(N_I, N_E+1, t)\right) + \delta u(N_I+1, N_E, t) \\
> \quad & s_E + f_E(N_E) + \delta\left(\Pi^E(N_I, N_E+1, t) - \Pi^E(N_I+1, N_E, t)\right) + \delta u(N_I, N_E+1, t),
\end{aligned}$$

so

$$\begin{aligned}
\Pi^I(N_I, N_E, t+1) &= \delta \Pi^I(N_I, N_E+1, t) \\
&\quad + \max\left\{ \begin{array}{c} s_I - s_E + f_I(N_I) - f_E(N_E) \\ +\delta \left(\begin{array}{c} \Pi^I(N_I+1, N_E, t) + \Pi^E(N_I+1, N_E, t) + u(N_I+1, N_E, t) \\ -\Pi^I(N_I, N_E+1, t) - \Pi^E(N_I, N_E+1, t) - u(N_I, N_E+1, t) \end{array} \right), 0 \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
\Pi^E(N_I, N_E, t+1) &= \delta \Pi^E(N_I+1, N_E, t) \\
&\quad + \max\left\{ \begin{array}{c} s_E - s_I + f_E(N_E) - f_I(N_I) \\ +\delta \left(\begin{array}{c} \Pi^I(N_I, N_E+1, t) + \Pi^E(N_I, N_E+1, t) + u(N_I, N_E+1, t) \\ -\Pi^I(N_I+1, N_E, t) - \Pi^E(N_I+1, N_E, t) - u(N_I+1, N_E, t) \end{array} \right), 0 \end{array} \right\}.
\end{aligned}$$

Using the induction hypothesis for the case with t periods left and states (N_I+1, N_E) and (N_I, N_E+1), straightforward calculations lead to

$$\begin{aligned}
\Pi^I(N_I, N_E, t+1) &= \max\left\{ \sum_{j=0}^{t} \delta^j \left(s_I + f_I(N_I+j) - s_E - f_E(N_E+j)\right), 0 \right\} \\
\Pi^E(N_I, N_E, t+1) &= \max\left\{ \sum_{j=0}^{t} \delta^j \left(s_E + f_E(N_E+j) - s_I - f_I(N_I+j)\right), 0 \right\}
\end{aligned}$$

$$\begin{aligned}
\Delta(N_I, N_E, t+1) &= \frac{\sum_{j=0}^{t} \delta^j \left(f_I(N_I+j) - f_E(N_E+j)\right)}{\sum_{j=0}^{t-1} \delta^j} \\
u(N_I, N_E, t+1) &= \min\left\{ \sum_{j=0}^{t} \delta^j \left(s_I - c + f_I(N_I+j)\right), \sum_{j=0}^{t} \delta^j \left(s_E - c + f_E(N_E+j)\right) \right\}
\end{aligned}$$

Thus, the result holds for $t+1$. By induction, we have thus proven the result for any $t \geq 1$.

C Proof of Proposition 10

We first treat the case with Pareto beliefs. In addition to proving that the cutoff $\Delta(N_I, N_E)$ takes the expression stated in Proposition 10 in the main text, we want to show the two firms' value functions are

$$V^I(N_I,N_E)=\begin{cases}\frac{s_I-s_E}{(1-\delta)^2}+\frac{\Delta(N_I,N_E)-\delta\Delta(N_I,N_E+1)}{(1-\delta)^3} & \text{if} \quad s_E-s_I<\Delta(N_I,N_E+1)\\ \frac{s_I-s_E+\Delta(N_I,N_E)}{(1-\delta)^3} & \text{if} \quad \Delta(N_I,N_E+1)\leq s_E-s_I<\Delta(N_I,N_E)\\ 0 & \text{if} \quad s_E-s_I\geq\Delta(N_I,N_E)\end{cases}$$

$$V^E(N_I,N_E)=\begin{cases}0 & \text{if} \quad s_E-s_I<\Delta(N_I,N_E)\\ \frac{s_E-s_I-\Delta(N_I,N_E)}{(1-\delta)^3} & \text{if} \quad \Delta(N_I,N_E)\leq s_E-s_I<\Delta(N_I+1,N_E)\\ \frac{s_E-s_I}{(1-\delta)^2}-\frac{\Delta(N_I,N_E)-\delta\Delta(N_I+1,N_E)}{(1-\delta)^3} & \text{if} \quad s_E-s_I\geq\Delta(N_I+1,N_E)\end{cases}.$$

Assume initially that consumers would never consider buying both products, an assumption we will relax at the end. Suppose to start with $N_I=\overline{N}_I$ and $N_E=\overline{N}_E$, i.e. both firms have reached their respective thresholds, in which case beliefs do not matter. Suppose I charges p^I and E charges p^E in the current period. Then because consumers care about the PDV of their stream of future utility, new consumers in the current period choose I iff

$$\frac{s_I+f_I(\overline{N}_I)}{1-\delta}-p^I>\frac{s_E+f_E(\overline{N}_E)}{1-\delta}-p^E.$$

Furthermore, since learning has been exhausted for both firms, there is no future benefit to a firm of attracting the current consumers, beyond the price that it collects. Thus, both I and E are only willing to price as low as c in order to win in the current period. Given Bertrand competition and Pareto beliefs, and the fact firms can sell to a new set of consumers every period in the same way, we have

$$\begin{aligned}V^I(\overline{N}_I,\overline{N}_E) &= \max\left\{\frac{s_I-s_E+f_I(\overline{N}_I)-f_E(\overline{N}_E)}{(1-\delta)^2},0\right\}\\ V^E(\overline{N}_I,\overline{N}_E) &= \max\left\{\frac{s_E-s_I+f_E(\overline{N}_E)-f_I(\overline{N}_I)}{(1-\delta)^2},0\right\},\end{aligned}$$

which implies $\Delta(\overline{N}_I,\overline{N}_E)=f_I(\overline{N}_I)-f_E(\overline{N}_E)$.

Suppose the state is $(\overline{N}_I,N_E)$ where $1\leq N_E\leq\overline{N}_E$.[1] We have

$$V^I(\overline{N}_I,N_E)=\begin{cases}\frac{s_I+f_I(\overline{N}_I)-s_E-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j)-\delta^{\overline{N}_E-N_E}f_E(\overline{N}_E)}{(1-\delta)^2} & \text{if} \quad s_E-s_I<\Delta(\overline{N}_I,N_E+1)\\ \frac{s_I-s_E+\Delta(\overline{N}_I,N_E)}{(1-\delta)^3} & \text{if} \quad \Delta(\overline{N}_I,N_E+1)\leq s_E-s_I<\Delta(\overline{N}_I,N_E)\\ 0 & \text{if} \quad s_E-s_I\geq\Delta(\overline{N}_I,N_E)\end{cases}$$

$$V^E(\overline{N}_I,N_E)=\begin{cases}0 & \text{if} \quad s_E-s_I<\Delta(\overline{N}_I,N_E)\\ \frac{s_E-s_I-\Delta(\overline{N}_I,N_E)}{(1-\delta)^2} & \text{if} \quad s_E-s_I\geq\Delta(\overline{N}_I,N_E)\end{cases}$$

[1] The expressions below go through for $N_E=\overline{N}_E$ with some slight abuse of notation.

$$\Delta(\overline{N}_I,N_E)=f_I(\overline{N}_I)-(1-\delta)^2\sum_{j=0}^{\overline{N}_E-N_E-1}(j+1)\delta^j f_E(N_E+j)-\delta^{\overline{N}_E-N_E}(\overline{N}_E-N_E+1-(\overline{N}_E-N_E)\delta)f_E(\overline{N}_E).$$

Consider now the state $(\overline{N}_I, N_E-1)$. Using the generalization of (9) from the main text to the current setting in which consumers get a lifetime of utility from a purchase decision, I wins if and only if

$$\begin{aligned}&\frac{s_I+f_I(\overline{N}_I)}{1-\delta}+\delta(V^I(\overline{N}_I,N_E-1)-V^I(\overline{N}_I,N_E))\\ >\ &\sum_{j=0}^{\overline{N}_E-N_E}\delta^j(s_E+f_E(N_E-1+j))+\delta^{\overline{N}_E-N_E+1}(\frac{s_E+f_E(\overline{N}_E)}{1-\delta})+\delta(V^E(\overline{N}_I,N_E)-V^E(\overline{N}_I,N_E-1)).\end{aligned}$$

Following the same logic as in the derivation of (10)-(11) in the main text, this implies

$$V^I(\overline{N}_I,N_E-1)=\delta V^I(\overline{N}_I,N_E)+\max\left\{\begin{array}{c}\frac{s_I+f_I(\overline{N}_I)-s_E-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j)-\delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)}{1-\delta}\\ +\delta\left(\begin{array}{c}V^I(\overline{N}_I,N_E-1)+V^E(\overline{N}_I,N_E-1)\\ -V^I(\overline{N}_I,N_E)-V^E(\overline{N}_I,N_E)\end{array}\right)\end{array},0\right\}$$

$$V^E(\overline{N}_I,N_E-1)=\delta V^E(\overline{N}_I,N_E-1)+\max\left\{\begin{array}{c}\frac{s_E+(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j)+\delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)-s_I-f_I(\overline{N}_I)}{1-\delta}\\ +\delta\left(\begin{array}{c}V^I(\overline{N}_I,N_E)+V^E(\overline{N}_I,N_E)\\ -V^I(\overline{N}_I,N_E-1)-V^E(\overline{N}_I,N_E-1)\end{array}\right)\end{array},0\right\}.$$

There are two cases. Suppose first $V^E(\overline{N}_I,N_E-1)=0$. Following the same steps as in the proof of Proposition 1 in the main text, we have

$$\begin{aligned}V^I(\overline{N}_I,N_E-1)\ &=\ \frac{s_I+f_I(\overline{N}_I)-s_E-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j)-\delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)}{(1-\delta)^2}\\ &\quad-\frac{\delta}{1-\delta}V^E(\overline{N}_I,N_E)\end{aligned}$$

$$=\begin{cases}\frac{s_I+f_I(\overline{N}_I)-s_E-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j)-\delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)}{(1-\delta)^2} & \text{if}\quad s_E-s_I<\Delta(\overline{N}_I,N_E)\\ \frac{s_I+f_I(\overline{N}_I)-s_E-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j)-\delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)}{(1-\delta)^2}-\frac{\delta}{1-\delta}\frac{s_E-s_I-\Delta(\overline{N}_I,N_E)}{(1-\delta)^2} & \text{if}\quad s_E-s_I\geq\Delta(\overline{N}_I,N_E)\end{cases}$$

thus establishing that

$$V^I(\overline{N}_I,N_E-1)=\begin{cases}\frac{s_I-s_E}{(1-\delta)^2}+\frac{\Delta(\overline{N}_I,N_E-1)-\delta\Delta(\overline{N}_I,N_E)}{(1-\delta)^3} & \text{if}\quad s_E-s_I<\Delta(\overline{N}_I,N_E)\\ \frac{s_I-s_E+\Delta(\overline{N}_I,N_E-1)}{(1-\delta)^3} & \text{if}\quad \Delta(\overline{N}_I,N_E)\leq s_E-s_I\leq\Delta(\overline{N}_I,N_E-1)\\ 0 & \text{if}\quad s_E-s_I\geq\Delta(\overline{N}_I,N_E-1)\end{cases},$$

where $\Delta(\overline{N}_I,N_E-1)$ is consistent with the general formula given in the proposition. Similarly, following the steps of Proposition 1 in the main text, if $V^E(\overline{N}_I, N_E - 1) > 0$ we obtain

$$V^I(\overline{N}_I, N_E - 1) = V^I(\overline{N}_I, N_E) = 0$$

$$\begin{aligned} V^E(\overline{N}_I, N_E - 1) &= \frac{s_E + (1-\delta)\sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E - 1 + j) + \delta^{\overline{N}_E - N_E + 1} f_E(\overline{N}_E) - s_I - f_I(\overline{N}_I)}{1-\delta} \\ &\quad + \delta V^E(\overline{N}_I, N_E) \end{aligned}$$

$$= \begin{cases} \frac{s_E - s_I - f_I(\overline{N}_I) + (1-\delta)\sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E - 1 + j) + \delta^{\overline{N}_E - N_E + 1} f_E(\overline{N}_E)}{1-\delta} & \text{if} \quad s_E - s_I < \Delta(\overline{N}_I, N_E) \\ \frac{\begin{array}{c} s_E - s_I - f_I(\overline{N}_I) + (1-\delta)^2 \sum_{j=0}^{\overline{N}_E - N_E} (j+1)\delta^j f_E(N_E - 1 + j) \\ + \delta^{\overline{N}_E - N_E + 1}(\overline{N}_E - N_E + 2 - (\overline{N}_E - N_E + 1)\delta) f_E(\overline{N}_E) \end{array}}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta(\overline{N}_I, N_E) \end{cases},$$

and so

$$V^E(\overline{N}_I, N_E - 1) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta(\overline{N}_I, N_E - 1) \\ \frac{s_E - s_I - \Delta(\overline{N}_I, N_E - 1)}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta(\overline{N}_I, N_E - 1) \end{cases}.$$

Thus, combining the two cases, we conclude that the result holds for the state $(\overline{N}_I, N_E - 1)$. By induction, this implies it holds for all states $(N_I = \overline{N}_I, N_E)$ with $0 \leq N_E \leq \overline{N}_E$. By symmetry, it also holds for any state $(N_I, N_E = \overline{N}_E)$ with $0 \leq N_I \leq \overline{N}_I$ with $m \geq 0$.

Consider now the case (N_I, N_E) with $0 \leq N_I \leq \overline{N}_I - 1$ and $0 \leq N_E \leq \overline{N}_E - 1$, and suppose result holds for $(N_I + 1, N_E)$ and $(N_I, N_E + 1)$. I wins the current period if and only if

$$\begin{aligned} &\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j (s_I + f_I(N_I + j)) + \delta^{\overline{N}_I - N_I}(\frac{s_I + f_I(\overline{N}_I)}{1-\delta}) + \delta(V^I(N_I + 1, N_E) - V^I(N_I, N_E + 1)) \\ > &\sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j (s_E + f_E(N_E + j)) + \delta^{\overline{N}_E - N_E}(\frac{s_E + f_E(\overline{N}_E)}{1-\delta}) + \delta(V^E(N_I, N_E + 1) - V^E(N_I + 1, N_E)). \end{aligned}$$

We then have

$$V^I(N_I, N_E) = \delta V^I(N_I, N_E{+}1) + \max \left\{ \begin{array}{c} \frac{s_I - s_E + \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{1-\delta} \\ + \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j) - \sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j) \\ + \delta \left(\begin{array}{c} V^I(N_I + 1, N_E) + V^E(N_I + 1, N_E) \\ - V^I(N_I, N_E + 1) - V^E(N_I, N_E + 1) \end{array} \right), 0 \end{array} \right\}$$

$$V^E(N_I, N_E) = \delta V^E(N_I{+}1, N_E) + \max \left\{ \begin{array}{c} \frac{s_E - s_I + \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{1-\delta} \\ + \sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j) - \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j (f_I(N_I + j)) \\ + \delta \left(\begin{array}{c} V^I(N_I, N_E + 1) + V^E(N_I, N_E + 1) \\ - V^I(N_I + 1, N_E) - V^E(N_I + 1, N_E) \end{array} \right), 0 \end{array} \right\}.$$

Two cases. Suppose first I wins the current period, so

$$V^E(N_I, N_E) = V^E(N_I + 1, N_E) = 0$$

and

$$\begin{aligned} V^I(N_I, N_E) &= \frac{s_I - s_E + \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{1-\delta} \\ &+ \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j) - \sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j) \\ &+ \delta(V^I(N_I + 1, N_E) - V^E(N_I, N_E + 1)) \end{aligned}$$

From the induction hypothesis, we have

$$V^I(N_I{+}1, N_E) = \begin{cases} \frac{s_I - s_E + \delta^{\overline{N}_I - N_I - 1}(\overline{N}_I - N_I - (\overline{N}_I - N_I - 1)\delta) f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{(1-\delta)^2} \\ \quad + \sum_{j=0}^{\overline{N}_I - N_I - 2} (j+1)\delta^j f_I(N_I + 1 + j) \\ \quad - \frac{\sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j)}{1-\delta} & \text{if } s_E - s_I < \Delta(N_I + 1, N_E + 1) \\ \frac{s_I - s_E + \Delta(N_I + 1, N_E)}{(1-\delta)^3} & \text{if } \Delta(N_I + 1, N_E + 1) \le s_E - s_I < \Delta(N_I + 1, N_E) \\ 0 & \text{if } s_E - s_I \ge \Delta(N_I + 1, N_E) \end{cases}$$

$$V^E(N_I, N_E{+}1) = \begin{cases} 0 & \text{if } s_E - s_I < \Delta(N_I, N_E + 1) \\ \frac{s_E - s_I - \Delta(N_I, N_E + 1)}{(1-\delta)^3} & \text{if } \Delta(N_I, N_E + 1) \le s_E - s_I < \Delta(N_I + 1, N_E + 1) \\ \frac{s_E - s_I + \delta^{\overline{N}_E - N_E - 1}(\overline{N}_E - N_E - (\overline{N}_E - N_E - 1)\delta) f_E(\overline{N}_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{(1-\delta)^2} \\ \quad + \sum_{j=0}^{\overline{N}_E - N_E - 2} (j+1)\delta^j f_E(N_E + 1 + j) \\ \quad - \frac{\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j)}{1-\delta} & \text{if } s_E - s_I \ge \Delta(N_I + 1, N_E + 1) \end{cases}$$

Plugging these two expressions into the expression of $V^I(N_I, N_E)$ above, we obtain

$$V^I(N_I, N_E) = \frac{s_I - s_E + \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{1-\delta} + \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I{+}j) - \sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E{+}j)$$

$$+\delta\begin{cases} \frac{s_I-s_E+\delta^{\overline{N}_I-N_I-1}(\overline{N}_I-N_I-(\overline{N}_I-N_I-1)\delta)f_I(\overline{N}_I)-\delta^{\overline{N}_E-N_E}f_E(\overline{N}_E)}{(1-\delta)^2} + \sum_{j=0}^{\overline{N}_I-N_I-2}(j+1)\delta^j f_I(N_I+1+j) - \frac{\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j)}{1-\delta} & \text{if } s_E-s_I < \Delta(N_I+1,N_E+1) \\ \frac{s_I-s_E+\delta^{\overline{N}_I-N_I-1}(\overline{N}_I-N_I-(\overline{N}_I-N_I-1)\delta)f_I(\overline{N}_I)-\delta^{\overline{N}_E-N_E}(\overline{N}_E-N_E+1-(\overline{N}_E-N_E)\delta)f_E(\overline{N}_E)}{(1-\delta)^3} + \frac{\sum_{j=0}^{\overline{N}_I-N_I-2}(j+1)\delta^j f_I(N_I+1+j)-\sum_{j=0}^{\overline{N}_E-N_E-1}(j+1)\delta^j f_E(N_E+j)}{1-\delta} & \text{if } \Delta(N_I+1,N_E+1) \le s_E-s_I < \Delta(N_I+1,N_E) \\ 0 & \text{if } s_E-s_I \ge \Delta(N_I+1,N_E) \end{cases}$$

$$-\delta\begin{cases} 0 & \text{if } s_E-s_I < \Delta(N_I,N_E+1) \\ \frac{s_E-s_I-\delta^{\overline{N}_I-N_I}(\overline{N}_I-N_I+1-(\overline{N}_I-N_I)\delta)f_I(\overline{N}_I)+\delta^{\overline{N}_E-N_E-1}(\overline{N}_E-N_E-(\overline{N}_E-N_E-1)\delta)f_E(\overline{N}_E)}{(1-\delta)^3} + \frac{\sum_{j=0}^{\overline{N}_E-N_E-2}(j+1)\delta^j f_E(N_E+1+j)-\sum_{j=0}^{\overline{N}_I-N_I-1}(j+1)\delta^j f_I(N_I+j)}{1-\delta} & \text{if } \Delta(N_I,N_E+1) \le s_E-s_I < \Delta(N_I+1,N_E+1) \\ \frac{s_E-s_I+\delta^{\overline{N}_E-N_E-1}(\overline{N}_E-N_E-(\overline{N}_E-N_E-1)\delta)f_E(\overline{N}_E)-\delta^{\overline{N}_I-N_I}f_I(\overline{N}_I)}{(1-\delta)^2} + \sum_{j=0}^{\overline{N}_E-N_E-2}(j+1)\delta^j f_E(N_E+1+j) - \frac{\sum_{j=0}^{\overline{N}_I-N_I-1}\delta^j f_I(N_I+j)}{1-\delta} & \text{if } s_E-s_I \ge \Delta(N_I+1,N_E+1) \end{cases}$$

Straightforward calculations and the imposition of the condition $V^I(N_I,N_E)\ge 0$ lead to

$$V^I(N_I,N_E)=\begin{cases} \frac{s_I-s_E+\delta^{\overline{N}_I-N_I}(\overline{N}_I-N_I+1-(\overline{N}_I-N_I)\delta)f_I(\overline{N}_I)-\delta^{\overline{N}_E-N_E}f_E(\overline{N}_E)}{(1-\delta)^2} + \sum_{j=0}^{\overline{N}_I-N_I-1}(j+1)\delta^j f_I(N_I+j) - \frac{\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j)}{1-\delta} & \text{if } s_E-s_I < \Delta(N_I,N_E+1) \\ \frac{s_I-s_E+\Delta(N_I,N_E)}{(1-\delta)^3} & \text{if } \Delta(N_I,N_E+1) \le s_E-s_I < \Delta(N_I,N_E) \\ 0 & \text{if } s_E-s_I \ge \Delta(N_I,N_E) \end{cases},$$

where

$$\Delta(N_I, N_E) = (1-\delta)^2 \sum_{j=0}^{\overline{N}_I - N_I - 1} (j+1)\delta^j f_I(N_I + j) + \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I)$$
$$-(1-\delta)^2 \sum_{j=0}^{\overline{N}_E - N_E - 1} (j+1)\delta^j f_E(N_E + j) - \delta^{\overline{N}_E - N_E}(\overline{N}_E - N_E + 1 - (\overline{N}_E - N_E)\delta) f_E(\overline{N}_E).$$

The case in which E wins the current period (so $V^I(N_I, N_E) = V^I(N_I, N_E + 1) = 0$) is symmetric, so we have $V^E(N_I, N_E) = V^E(N_I + 1, N_E) = 0$ and

$$V^E(N_I, N_E) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta(N_I, N_E) \\ \frac{s_E - s_I - \Delta(N_I, N_E)}{(1-\delta)^3} & \text{if} \quad \begin{array}{c}\Delta(N_I, N_E) \leq s_E - s_I \\ < \Delta(N_I + 1, N_E)\end{array} \\ \begin{array}{c} \frac{s_E - s_I + \delta^{\overline{N}_E - N_E}(\overline{N}_E - N_E + 1 - (\overline{N}_E - N_E)\delta) f_E(\overline{N}_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{(1-\delta)^2} \\ + \sum_{j=0}^{\overline{N}_E - N_E - 1}(j+1)\delta^j f_E(N_E + j) \\ - \frac{\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j)}{1-\delta} \end{array} & \text{if} \quad s_E - s_I \geq \Delta(N_I + 1, N_E) \end{cases}$$

Thus, the result holds for the state (N_I, N_E). Using this induction reasoning repeatedly, we conclude that the result holds for every state (N_I, N_E) with $0 \leq N_I \leq \overline{N}_I$ and $0 \leq N_E \leq \overline{N}_E$.

Finally, along the equilibrium path we have just determined, in any given period, "old" consumers (i.e. those that arrived in previous periods) do not want to switch and also purchase from the losing firm. This is because consumers that arrive in the current period do not purchase from the losing firm in equilibrium. Indeed, for any given prices charged by the two firms (including equilibrium prices), old consumers are less willing to buy from the losing firm than current consumers, because the former pay nothing to keep consuming from the winning firm. The only exception to this logic is if prices are ever negative, which we ruled out by assumption in the setup of our model.

To see that the outcome with Pareto beliefs is socially efficient, compare the total PDVs of utility created by the two firms for all current and future consumers. E creates more total utility if and only if

$$\sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j \Big(\sum_{k=0}^{\overline{N}_E - N_E - 1 - j} \delta^k (s_E + f_E(N_E + j + k)) + \delta^{\overline{N}_E - N_E - j} \frac{s_E + f_E(\overline{N}_E)}{1-\delta} \Big) + \delta^{\overline{N}_E - N_E} \frac{s_E + f_E(\overline{N}_E)}{(1-\delta)^2}$$
$$\geq \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j \Big(\sum_{k=0}^{\overline{N}_I - N_I - 1 - j} \delta^k (s_I + f_I(N_I + j + k)) + \delta^{\overline{N}_I - N_I - j} \frac{s_I + f_I(\overline{N}_I)}{1-\delta} \Big) + \delta^{\overline{N}_I - N_I} \frac{s_I + f_I(\overline{N}_I)}{(1-\delta)^2}.$$

Simple calculations show that this is equivalent to

$$\frac{s_E + (1-\delta)^2 \sum_{j=0}^{\overline{N}_E - N_E - 1}(j+1)\delta^j f_E(N_E+j) + \delta^{\overline{N}_E - N_E}(\overline{N}_E - N_E + 1 - (\overline{N}_E - N_E)\delta) f_E(\overline{N}_E)}{(1-\delta)^2}$$
$$\geq \quad \frac{s_I + (1-\delta)^2 \sum_{j=0}^{\overline{N}_I - N_I - 1}(j+1)\delta^j f_I(N_I+j) + \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I)}{(1-\delta)^2}$$

which in turn is equivalent to $s_E - s_I \geq \Delta(N_I, N_E)$.

Consider now the case with beliefs that favor I. In this case, in addition to establishing the cutoff $\Delta^I(N_I, N_E)$ defined in the proposition, we want to show the two firms' value functions are

$$V^I(N_I, N_E) = \begin{cases} \frac{s_I - s_E}{(1-\delta)^2} + \frac{\Delta^I(N_I,N_E) - \delta\Delta^I(N_I,N_E+1)}{(1-\delta)^3} & \text{if} \quad s_E - s_I < \Delta^I(N_I, N_E+1) \\ \frac{s_I - s_E + \Delta^I(N_I,N_E)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I, N_E+1) \leq s_E - s_I < \Delta^I(N_I, N_E) \\ 0 & \text{if} \quad s_E - s_I \geq \Delta^I(N_I, N_E) \end{cases},$$

$$V^E(N_I, N_E) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(N_I, N_E) \\ \frac{s_E - s_I - \Delta(N_I,N_E)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I, N_E) \leq s_E - s_I < \Delta^I(N_I+1, N_E) \\ \frac{s_E - s_I}{(1-\delta)^2} - \frac{\Delta^I(N_I,N_E) - \delta\Delta^I(N_I+1,N_E)}{(1-\delta)^3} & \text{if} \quad s_E - s_I \geq \Delta^I(N_I+1, N_E) \end{cases}.$$

Nothing changes for the state $(N_I, N_E) = (\overline{N}_I, \overline{N}_E)$, so $\Delta^I(\overline{N}_I, \overline{N}_E) = f_I(\overline{N}_I) - f_E(\overline{N}_E)$ and $V^I(\overline{N}_I, \overline{N}_E)$ and $V^E(\overline{N}_I, \overline{N}_E)$ are defined as above. Suppose the result holds for the state $(\overline{N}_I, N_E)$ with $1 \leq N_E \leq \overline{N}_E$, i.e.

$$\Delta^I(\overline{N}_I, N_E) = f_I(\overline{N}_I) - (1-\delta)\sum_{j=0}^{\overline{N}_E - N_E - 1}\delta^j f_E(N_E+j) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)$$

$$V^I(\overline{N}_I, N_E) = \begin{cases} \frac{s_I - s_E + f_I(\overline{N}_I) - f_E(N_E)}{(1-\delta)^2} & \text{if} \quad s_E - s_I < \Delta^I(\overline{N}_I, N_E+1) \\ \frac{s_I - s_E + f_I(\overline{N}_I) - (1-\delta)\sum_{j=0}^{\overline{N}_E - N_E - 1}\delta^j f_E(N_E+j) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{(1-\delta)^3} & \text{if} \quad \begin{array}{c}\Delta^I(\overline{N}_I, N_E+1) \leq s_E - s_I \\ < \Delta^I(\overline{N}_I, N_E)\end{array} \\ 0 & \text{if} \quad s_E - s_I \geq \Delta^I(\overline{N}_I, N_E) \end{cases}$$

$$V^E(\overline{N}_I, N_E) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(\overline{N}_I, N_E) \\ \frac{s_E - s_I + (1-\delta)\sum_{j=0}^{\overline{N}_E - N_E - 1}\delta^j f_E(N_E+j) + \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E) - f_I(\overline{N}_I)}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta^I(\overline{N}_I, N_E) \end{cases}.$$

Now consider the state $(\overline{N}_I, N_E - 1)$. Favorable beliefs for I imply I wins the current period iff

$$\frac{s_I + f_I(\overline{N}_I)}{1-\delta} + \delta(V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E)) > \frac{s_E + f_E(N_E - 1)}{1-\delta} + \delta(V^E(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1)).$$

And following the same steps as above, we have

$$V^I(\overline{N}_I, N_E - 1) = \delta V^I(\overline{N}_I, N_E) + \max \left\{ \begin{array}{c} \frac{s_I + f_I(\overline{N}_I) - s_E - f_E(N_E - 1)}{1-\delta} \\ +\delta \left(\begin{array}{c} V^I(\overline{N}_I, N_E - 1) + V^E(\overline{N}_I, N_E - 1) \\ -V^I(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E) \end{array} \right), 0 \end{array} \right\}$$

$$V^E(\overline{N}_I, N_E - 1) = \delta V^E(\overline{N}_I, N_E - 1) + \max \left\{ \begin{array}{c} \frac{s_E + f_E(N_E - 1) - s_I - f_I(\overline{N}_I)}{1-\delta} \\ +\delta \left(\begin{array}{c} V^I(\overline{N}_I, N_E) + V^E(\overline{N}_I, N_E) \\ -V^I(\overline{N}_I, N_E - 1) - V^E(\overline{N}_I, N_E - 1) \end{array} \right), 0 \end{array} \right\}.$$

There are two possibilities: $V^E(\overline{N}_I, N_E - 1) = 0$ and $V^E(\overline{N}_I, N_E - 1) > 0$.

Suppose first $V^E(\overline{N}_I, N_E - 1) = 0$, so $V^I(\overline{N}_I, N_E - 1) \geq \delta V^I(\overline{N}_I, N_E)$ and

$$\begin{aligned} V^I(\overline{N}_I, N_E - 1) &= \frac{s_I + f_I(\overline{N}_I) - s_E - f_E(N_E - 1)}{(1-\delta)^2} - \frac{\delta}{1-\delta} V^E(\overline{N}_I, N_E) \\ &= \begin{cases} \frac{s_I - s_E + f_I(\overline{N}_I) - f_E(N_E - 1)}{(1-\delta)^2} & \text{if} \quad s_E - s_I < \Delta^I(\overline{N}_I, N_E) \\ \frac{s_I - s_E - (1-\delta)\sum_{j=0}^{\overline{N}_E - N_E} \delta^j f_E(N_E - 1 + j) - \delta^{\overline{N}_E - N_E + 1} f_E(\overline{N}_E) + f_I(\overline{N}_I)}{(1-\delta)^3} & \text{if} \quad s_E - s_I \geq \Delta^I(\overline{N}_I, N_E) \end{cases}. \end{aligned}$$

Note that when $s_E - s_I < \Delta^I(\overline{N}_I, N_E)$, we have

$$\frac{s_I - s_E + f_I(\overline{N}_I) - f_E(N_E - 1)}{(1-\delta)^2} > \frac{s_I - s_E + f_I(\overline{N}_I) - f_E(N_E)}{(1-\delta)^2} = V^I(\overline{N}_I, N_E) > \delta V^I(\overline{N}_I, N_E)$$

and when $s_E - s_I \geq \Delta^I(\overline{N}_I, N_E)$, we have $V^I(\overline{N}_I, N_E) = 0$, so the binding constraint on this range must be $V^I(\overline{N}_I, N_E - 1) \geq 0$, which is equivalent to $s_E - s_I \leq \Delta^I(\overline{N}_I, N_E - 1)$.

Next, suppose $V^E(\overline{N}_I, N_E - 1) > 0$, so $V^I(\overline{N}_I, N_E - 1) = \delta V^I(\overline{N}_I, N_E)$. Since $V^I(\overline{N}_I, N_E - 1) \geq V^I(\overline{N}_I, N_E)$ and $\delta < 1$, this implies $V^I(\overline{N}_I, N_E - 1) = V^I(\overline{N}_I, N_E) = 0$, so

$$\begin{aligned} V^E(\overline{N}_I, N_E - 1) &= \frac{s_E + \delta f_E(N_E - 1) - s_I - f_I(\overline{N}_I)}{1-\delta} + \delta V^E(\overline{N}_I, N_E) \\ &= \begin{cases} \frac{s_E - s_I + f_E(N_E - 1) - f_I(\overline{N}_I)}{1-\delta} & \text{if} \quad s_E - s_I < \Delta^I(\overline{N}_I, N_E) \\ \frac{s_E - s_I - \Delta^I(\overline{N}_I, N_E - 1)}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta^I(\overline{N}_I, N_E) \end{cases}. \end{aligned}$$

Note that $s_E - s_I + f_E(N_E - 1) - f_I(\overline{N}_I) < s_E - s_I + f_E(N_E) - f_I(\overline{N}_I) < 0$ given $s_E - s_I < \Delta^I(\overline{N}_I, N_E)$, and $s_E - s_I - \Delta^I(\overline{N}_I, N_E - 1) > 0$ if and only if $s_E - s_I \geq \Delta^I(\overline{N}_I, N_E - 1)$.

Thus, we have proven

$$V^I(\overline{N}_I, N_E - 1) = \begin{cases} \frac{s_I - s_E + f_I(\overline{N}_I) - f_E(N_E - 1)}{(1-\delta)^2} & \text{if} \quad s_E - s_I < \Delta^I(\overline{N}_I, N_E) \\ \frac{s_I - s_E + \Delta^I(\overline{N}_I, N_E - 1)}{(1-\delta)^3} & \text{if} \quad \Delta^I(\overline{N}_I, N_E) \leq s_E - s_I < \Delta^I(\overline{N}_I, N_E - 1) \\ 0 & \text{if} \quad s_E - s_I \geq \Delta^I(\overline{N}_I, N_E - 1) \end{cases}.$$

$$V^E(\overline{N}_I, N_E-1) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(\overline{N}_I, N_E-1) \\ \frac{s_E - s_I - \Delta^I(\overline{N}_I, N_E-1)}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta^I(\overline{N}_I, N_E-1) \end{cases}.$$

So the result holds for the state $(\overline{N}_I, N_E-1)$. By repeated application of the induction hypothesis, the result holds for all states $(\overline{N}_I, N_E)$, with $0 \leq N_E \leq \overline{N}_E$.

For any state $(N_I, \overline{N}_E)$ with $0 \leq N_I \leq \overline{N}_I$, the outcome is the same as under Pareto expectations because E is already at its threshold, so expectations don't affect it. This means we already know we have

$$\Delta(N_I, \overline{N}_E) = \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I) - f_E(\overline{N}_E) + (1-\delta)^2 \sum_{j=0}^{\overline{N}_I - N_I - 1} (j+1)\delta^j f_I(N_I + j)$$

$$V^I(N_I, \overline{N}_E) = \begin{cases} \dfrac{\begin{array}{c} s_I - s_E + (1-\delta)^2 \sum_{j=0}^{\overline{N}_I - N_I - 1} (j+1)\delta^j f_I(N_I + j) \\ +\delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I) - f_E(\overline{N}_E) \end{array}}{(1-\delta)^2} & \text{if} \quad s_E - s_I < \Delta^I(N_I, \overline{N}_E) \\ 0 & \text{if} \quad s_E - s_I \geq \Delta^I(N_I, \overline{N}_E) \end{cases}$$

$$V^E(N_I, \overline{N}_E) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(N_I, \overline{N}_E) \\ \frac{s_E - s_I - \Delta(N_I, \overline{N}_E)}{(1-\delta)^3} & \text{if} \quad \begin{array}{c} \Delta^I(N_I, \overline{N}_E) \leq s_E - s_I \\ < \Delta^I(N_I + 1, \overline{N}_E) \end{array} \\ \frac{s_E - s_I + f_E(\overline{N}_E) - (1-\delta)\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta^I(N_I + 1, \overline{N}_E) \end{cases},$$

which means the result holds for all states $(N_I, \overline{N}_E)$, with $0 \leq N_I \leq \overline{N}_I$.

Consider now the state (N_I, N_E) with $0 \leq N_I \leq \overline{N}_I - 1$ and $0 \leq N_E \leq \overline{N}_E - 1$, and suppose the result we want to prove holds for states $(N_I + 1, N_E)$ and $(N_I, N_E + 1)$. I wins the current period if and only if

$$\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j (s_I + f_I(N_I + j)) + \delta^{\overline{N}_I - N_I}\left(\frac{s_I + f_I(\overline{N}_I)}{1-\delta}\right) + \delta(V^I(N_I + 1, N_E) - V^I(N_I, N_E + 1))$$
$$> \frac{s_E + f_E(N_E)}{1-\delta} + \delta(V^E(N_I, N_E + 1) - V^E(N_I + 1, N_E)).$$

We then have

$$V^I(N_I, N_E) = \delta V^I(N_I, N_E + 1) + \max\left\{ \begin{array}{c} \frac{s_I - s_E + (1-\delta)\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j) + \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - f_E(N_E)}{1-\delta} \\ +\delta \left(\begin{array}{c} V^I(N_I + 1, N_E) + V^E(N_I + 1, N_E) \\ -V^I(N_I, N_E + 1) - V^E(N_I, N_E + 1) \end{array} \right), 0 \end{array} \right\}$$

$$V^E(N_I, N_E) = \delta V^E(N_I + 1, N_E) + \max\left\{ \begin{array}{c} \frac{s_E - s_I + f_E(N_E) - (1-\delta)\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j (f_I(N_I + j)) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{1-\delta} \\ +\delta \left(\begin{array}{c} V^I(N_I, N_E + 1) + V^E(N_I, N_E + 1) \\ -V^I(N_I + 1, N_E) - V^E(N_I + 1, N_E) \end{array} \right), 0 \end{array} \right\}.$$

There are two cases. Suppose first I wins the current period, so $V^E(N_I,N_E) = V^E(N_I+1,N_E) = 0$ and

$$V^I(N_I,N_E) = \frac{s_I - s_E + (1-\delta)\sum_{j=0}^{\overline{N}_I-N_I-1}\delta^j f_I(N_I+j) + \delta^{\overline{N}_I-N_I} f_I(\overline{N}_I) - f_E(N_E)}{1-\delta} + \delta(V^I(N_I+1,N_E) - V^E(N_I,N_E+1))$$

From the induction hypothesis, we have

$$V^I(N_I{+}1,N_E) = \begin{cases} \frac{s_I-s_E+\delta^{\overline{N}_I-N_I-1}(\overline{N}_I-N_I-(\overline{N}_I-N_I-1)\delta)f_I(\overline{N}_I)-f_E(N_E)}{(1-\delta)^2} + \sum_{j=0}^{\overline{N}_I-N_I-2}(j+1)\delta^j f_I(N_I+1+j) & \text{if} \quad s_E - s_I < \Delta^I(N_I+1,N_E+1) \\ \frac{s_I-s_E+\Delta^I(N_I+1,N_E)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I+1,N_E+1) \le s_E - s_I < \Delta^I(N_I+1,N_E) \\ 0 & \text{if} \quad s_E - s_I \ge \Delta^I(N_I+1,N_E) \end{cases}$$

$$V^E(N_I,N_E{+}1) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(N_I,N_E+1) \\ \frac{s_E-s_I-\Delta^I(N_I,N_E+1)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I,N_E+1) \le s_E - s_I < \Delta^I(N_I+1,N_E+1) \\ \frac{s_E-s_I+\delta^{\overline{N}_E-N_E-1}f_E(\overline{N}_E)-\delta^{\overline{N}_I-N_I}f_I(\overline{N}_I)}{(1-\delta)^2} + \frac{\sum_{j=0}^{\overline{N}_E-N_E-2}\delta^j f_E(N_E+1+j)-\sum_{j=0}^{\overline{N}_I-N_I-1}\delta^j f_I(N_I+j)}{1-\delta} & \text{if} \quad s_E - s_I \ge \Delta^I(N_I+1,N_E+1) \end{cases}.$$

Plugging these two expressions into the expression of $V^I(N_I,N_E)$ above, we obtain

$$V^I(N_I,N_E) = \frac{s_I - s_E + (1-\delta)\sum_{j=0}^{\overline{N}_I-N_I-1}\delta^j f_I(N_I+j) + \delta^{\overline{N}_I-N_I} f_I(\overline{N}_I) - f_E(N_E)}{1-\delta}$$

$$+\delta \begin{cases} \frac{s_I-s_E+\delta^{\overline{N}_I-N_I-1}(\overline{N}_I-N_I-(\overline{N}_I-N_I-1)\delta)f_I(\overline{N}_I)-f_E(N_E)}{(1-\delta)^2} + \sum_{j=0}^{\overline{N}_I-N_I-2}(j+1)\delta^j f_I(N_I+1+j) & \text{if} \quad s_E - s_I < \Delta^I(N_I+1,N_E+1) \\ \frac{s_I-s_E+\delta^{\overline{N}_I-N_I-1}(\overline{N}_I-N_I-(\overline{N}_I-N_I-1)\delta)f_I(\overline{N}_I)-\delta^{\overline{N}_E-N_E}f_E(\overline{N}_E)}{(1-\delta)^3} + \frac{\sum_{j=0}^{\overline{N}_I-N_I-2}(j+1)\delta^j f_I(N_I+1+j)}{1-\delta} - \frac{\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j)}{(1-\delta)^2} & \text{if} \quad \Delta^I(N_I+1,N_E+1) \le s_E - s_I < \Delta^I(N_I+1,N_E) \\ 0 & \text{if} \quad s_E - s_I \ge \Delta^I(N_I+1,N_E) \end{cases}$$

$$
-\delta \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(N_I, N_E+1) \\ \frac{s_E - s_I - \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I) + \delta^{\overline{N}_E - N_E - 1} f_E(\overline{N}_E)}{(1-\delta)^3} \\ \quad - \frac{\sum_{j=0}^{\overline{N}_E - N_E - 1} (j+1)\delta^j f_I(N_I + j)}{1-\delta} & \text{if} \quad \Delta^I(N_I, N_E+1) \leq s_E - s_I \\ \quad + \frac{\sum_{j=0}^{n-2} \delta^j f_E(N_E + 1 + j)}{(1-\delta)^2} & \quad < \Delta^I(N_I+1, N_E+1) \\ \frac{s_E - s_I + \delta^{\overline{N}_E - N_E - 1} f_E(\overline{N}_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{(1-\delta)^2} \\ \quad + \frac{\sum_{j=0}^{\overline{N}_E - N_E - 2} \delta^j f_E(N_E + 1 + j) - \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j)}{1-\delta} & \text{if} \quad s_E - s_I \geq \Delta^I(N_I+1, N_E+1) \end{cases}
$$

Straightforward calculations and the imposition of the condition $V^I(N_I, N_E) \geq 0$ lead to

$$
V^I(N_I, N_E) = \begin{cases} \frac{s_I - s_E + \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I) - f_E(N_E)}{(1-\delta)^2} \\ \quad + \sum_{j=0}^{\overline{N}_I - N_I - 1} (j+1)\delta^j f_I(N_I + j) & \text{if} \quad s_E - s_I < \Delta^I(N_I, N_E+1) \\ \frac{s_I - s_E + \Delta^I(N_I, N_E)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I, N_E+1) \leq s_E - s_I \\ & \quad < \Delta^I(N_I, N_E) \\ 0 & \text{if} \quad s_E - s_I \geq \Delta^I(N_I, N_E) \end{cases},
$$

where

$$
\begin{aligned}
\Delta^I(N_I, N_E) &= (1-\delta)^2 \sum_{j=0}^{\overline{N}_I - N_I - 1} (j+1)\delta^j f_I(N_I + j) + \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I) \\
&\quad -(1-\delta) \sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)
\end{aligned}
$$

Now suppose E wins the current period, so $V^I(N_I, N_E) = V^I(N_I, N_E+1) = 0$ and

$$
\begin{aligned}
V^E(N_I, N_E) &= \frac{s_E - s_I + f_E(N_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - (1-\delta) \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j)}{1-\delta} \\
&\quad + \delta(V^E(N_I, N_E+1) - V^I(N_I+1, N_E)).
\end{aligned}
$$

We already have the expressions of $V^I(N_I+1, N_E)$ and $V^E(N_I, N_E+1)$ from the induction hypothesis above. Plugging these two expressions into the expression of $V^E(N_I, N_E)$, we obtain

$$
V^E(N_I, N_E) = \frac{s_E - s_I + f_E(N_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - (1-\delta) \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j)}{1-\delta}
$$

$$
-\delta \begin{cases} \frac{s_I - s_E + \delta^{\overline{N}_I - N_I - 1}(\overline{N}_I - N_I - (\overline{N}_I - N_I - 1)\delta) f_I(\overline{N}_I) - f_E(N_E)}{(1-\delta)^2} + \sum_{j=0}^{\overline{N}_I - N_I - 2}(j+1)\delta^j f_I(N_I + 1 + j) & \text{if} \quad s_E - s_I < \Delta^I(N_I+1, N_E+1) \\ \frac{s_I - s_E + \delta^{\overline{N}_I - N_I - 1}(\overline{N}_I - N_I - (\overline{N}_I - N_I - 1)\delta) f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{(1-\delta)^3} + \frac{\sum_{j=0}^{\overline{N}_I - N_I - 2}(j+1)\delta^j f_I(N_I+1+j)}{1-\delta} - \frac{\sum_{j=0}^{\overline{N}_E - N_E - 1}\delta^j f_E(N_E+j)}{(1-\delta)^2} & \text{if} \quad \Delta^I(N_I+1, N_E+1) \le s_E - s_I < \Delta^I(N_I+1, N_E) \\ 0 & \text{if} \quad s_E - s_I \ge \Delta^I(N_I+1, N_E) \end{cases}
$$

$$
+\delta \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(N_I, N_E+1) \\ \frac{s_E - s_I - \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I) + \delta^{\overline{N}_E - N_E - 1} f_E(\overline{N}_E)}{(1-\delta)^3} - \frac{\sum_{j=0}^{\overline{N}_I - N_I - 1}(j+1)\delta^j f_I(N_I+j)}{1-\delta} + \frac{\sum_{j=0}^{\overline{N}_E - N_E - 2}\delta^j f_E(N_E+1+j)}{(1-\delta)^2} & \text{if} \quad \Delta^I(N_I, N_E+1) \le s_E - s_I < \Delta^I(N_I+1, N_E+1) \\ \frac{s_E - s_I + \delta^{\overline{N}_E - N_E - 1} f_E(\overline{N}_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{(1-\delta)^2} + \frac{\sum_{j=0}^{\overline{N}_E - N_E - 2}\delta^j f_E(N_E+1+j) - (1-\delta)\sum_{j=0}^{\overline{N}_I - N_I - 1}\delta^j f_I(N_I+j)}{1-\delta} & \text{if} \quad s_E - s_I \ge \Delta^I(N_I+1, N_E+1) \end{cases}
$$

Straightforward calculations and the imposition of the condition $V^E(N_I, N_E) \ge 0$ lead to

$$
V^E(N_I, N_E) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(N_I, N_E) \\ \frac{s_E - s_I - \Delta(N_I, N_E)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I, N_E) \le s_E - s_I < \Delta^I(N_I+1, N_E) \\ \frac{s_E - s_I + \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{(1-\delta)^2} + \frac{\sum_{j=0}^{\overline{N}_E - N_E - 1}\delta^j f_E(N_E+j) - \sum_{j=0}^{\overline{N}_I - N_I - 1}\delta^j f_I(N_I+j)}{1-\delta} & \text{if} \quad s_E - s_I \ge \Delta^I(N_I+1, N_E) \end{cases} .
$$

Thus, the result holds for the state (N_I, N_E). Using this induction reasoning repeatedly, we conclude that the result holds for every state (N_I, N_E) with $0 \le N_I \le \overline{N}_I$ and $0 \le N_E \le \overline{N}_E$.

The same reasoning as in the case with Pareto beliefs ensure that consumers do not also buy from the losing firm.

Finally, we wish to prove that $\Delta^I(N_I, N_E) > \Delta(N_I, N_E)$. This is equivalent to

$$
(1-\delta)^2 \sum_{j=0}^{\overline{N}_E - N_E - 1}(j+1)\delta^j f_E(N_E+j) + (\overline{N}_E - N_E)\delta^{\overline{N}_E - N_E}(1-\delta) f_E(\overline{N}_E) - (1-\delta) \sum_{j=0}^{\overline{N}_E - N_E - 1}\delta^j f_E(N_E+j) > 0.
$$

Dividing this through by $(1-\delta)$ and cancelling out some terms, we can rewrite the inequality as

$$\sum_{j=1}^{\overline{N}_E - N_E} (j\delta^j (f_E(N_E+j) - f_E(N_E+j-1))) > 0$$

which is true given $f(.)$ is increasing (indeed, strictly increasing for at least one step). Thus, we have proven that $\Delta^I(N_I, N_E) > \Delta(N_I, N_E)$.

D Generalization of analysis from Section 5.3

In this section, we analyze a generalization of the model that combines across-user and within-user learning from Section 5.3. We do so by allowing the two learning functions to be different for the two firms: they are $f_i^A(.)$ and $f_i^W(.)$ for firm $i = I, E$. In the main text, we have assumed for simplicity that $f_I^A(.) = f_E^A(.) = f^A(.)$ and $f_I^W(.) = f_E^W(.) = f^W(.)$.

First, suppose both firms have reached their respective thresholds in both types of learning, so they offer utilities $s_I + f_I^A(1) + f_I^W(1)$ and $s_E + f_E^A(1) + f_E^W(1)$ respectively. In this case, a consumer's decision in the current period (holding all other consumers' decisions fixed) has no impact on her options in future periods, so all consumers choose I in every period if and only if

$$s_I + f_I^A(1) + f_I^W(1) > s_E + f_E^A(1) + f_E^W(1).$$

Otherwise, all consumers choose E in every period. In this case, because both firms have reached their learning thresholds, there are no network effects and beliefs are irrelevant.

Next, suppose I has reached its thresholds in both types of learning, so it offers utility $s_I + f_I^A(1) + f_I^W(1)$, while E has not served any consumers in the past, so it offers $s_E + f_E^A(0) + f_E^W(0) = s_E$ in the current period. In this case, the only two possible pure strategy equilibria are as follows:

- All consumers choose I in every period. This is an equilibrium if and only if no individual consumer has an incentive to deviate. Consider the best possible deviation by an individual consumer. If that deviation involves adopting E in any period, then from the next period onwards, the consumer always chooses E if $s_E + f_E^W(1) \geq s_I + f_I^A(1) + f_I^W(1)$, or always chooses I if $s_E + f_E^W(1) < s_I + f_I^A(1) + f_I^W(1)$. If $s_E + f_E^W(1) < s_I + f_I^A(1) + f_I^W(1)$, then deviating to choosing E in any period cannot be profitable. If $s_E + f_E^W(1) \geq s_I + f_I^A(1) + f_I^W(1)$, then the best deviation is to choose E from the first period onwards. Thus, no individual consumer has an incentive to deviate if and only if

$$\frac{s_I + f_I^A(1) + f_I^W(1)}{1-\delta} > \frac{s_E + \delta f_E^W(1)}{1-\delta},$$

  i.e. if and only if

$$s_I + f_I^A(1) + f_I^W(1) > s_E + \delta f_E^W(1).$$

- All consumers choose E in every period. This is an equilibrium if and only if no individual consumer has an incentive to deviate. Consider the best possible deviation by an individual consumer. First, that deviation cannot involve any switching from the second period onwards. To see this, suppose the deviating consumer chooses E in the second period. Then from the third period onwards, the consumer always chooses E if $s_E + f_E^A(1) + f_E^W(1) \geq s_I + f_I^A(1) + f_I^W(1)$, or always chooses I if $s_E + f_E^A(1) + f_E^W(1) < s_I + f_I^A(1) + f_I^W(1)$. But if $s_E + f_E^A(1) + f_E^W(1) < s_I + f_I^A(1) + f_I^W(1)$, then the deviating consumer would have done better to choose I in every period. Now suppose the deviating consumer chooses I in the second period. If this deviation ever involves choosing E from the third period onwards, then it must be that $s_E + f_E^A(1) + f_E^W(1) \geq s_I + f_I^A(1) + f_I^W(1)$. But in that case the consumer would have done better to choose E at least from the second period onwards. Thus, there are only three possible deviations to consider, so all consumers choosing E is an equilibrium if and only if

$$\frac{s_E + \delta\left(f_E^A(1) + f_E^W(1)\right)}{1-\delta} \geq \max\left\{\begin{array}{c} s_E + \delta\frac{s_I + f_I^A(1) + f_I^W(1)}{1-\delta}, \frac{s_I + f_I^A(1) + f_I^W(1)}{1-\delta}, \\ s_I + f_I^A(1) + f_I^W(1) + \delta\frac{s_E + f_E^A(1) + \delta f_E^W(1)}{1-\delta} \end{array}\right\}.$$

  It is easily verified that this is equivalent to

$$s_E + \delta f_E^W(1) \geq s_I + f_I^A(1) + f_I^W(1).$$

Bottomline:

- if $s_E + \delta f_E^W(1) \geq s_I + f_I^A(1) + f_I^W(1)$, then the unique equilibrium is for all consumers to choose E in all periods.
- if $s_E + \delta f_E^W(1) < s_I + f_I^A(1) + f_I^W(1)$, then the unique equilibrium is for all consumers to choose I in all periods.

Finally, consider the case in which both firms start with no previous learning, i.e. $N_I = N_E = n_I = n_E = 0$. Again, there are only two possible pure strategy equilibria. To see this, suppose the equilibrium involves consumers choosing I in period 1 and then E in some later period (the case in which consumers choose E in period 1 and I in some later period is symmetric). After period 1, the utility offered by I to any consumer is fixed and equal to $s_I + f_I^A(1) + f_I^W(1)$. And after the first period in which consumers choose E, the utility offered by E to any consumer is fixed and equal to $s_E + f_E^A(1) + f_E^W(1)$. If $s_E + f_E^A(1) + f_E^W(1) < s_I + f_I^A(1) + f_I^W(1)$, then this cannot be an equilibrium because any consumer would be better off never choosing E and sticking to I after the first period We must therefore have $s_E + f_E^A(1) + f_E^W(1) > s_I + f_I^A(1) + f_I^W(1)$, so if consumers choose E in one period, then in equilibrium they must choose E forever after that period. In order for a consumer not to want to deviate and choose I for one more period, we must also have

$$s_I + f_I^A(1) + f_I^W(1) + \delta\left(s_E + f_E^A(1)\right) + \delta^2 \frac{s_E + f_E^A(1) + f_E^W(1)}{1-\delta} \leq s_E + \delta \frac{s_E + f_E^A(1) + f_E^W(1)}{1-\delta},$$

i.e.

$$s_I + f_I^A(1) + f_I^W(1) \leq s_E + \delta f_E^W(1),$$

which is stricter than $s_E + f_E^A(1) + f_E^W(1) > s_I + f_I^A(1) + f_I^W(1)$.

On the other hand, we also need to impose that a consumer does not want to deviate and choose E one period earlier. If the equilibrium has consumers choosing E from period 2, then this requires

$$s_E + \delta\left(s_E + f_E^W(1)\right) + \delta^2 \frac{s_E + f_E^A(1) + f_E^W(1)}{1-\delta} < s_I + \delta s_E + \delta^2 \frac{s_E + f_E^A(1) + f_E^W(1)}{1-\delta},$$

i.e.

$$s_E + \delta f_E^W(1) < s_I.$$

If the equilibrium has consumers choosing E from period 3 or later, then this requires

$$s_E + \delta\left(s_E + f_E^W(1)\right) + \delta^2 \frac{s_E + f_E^A(1) + f_E^W(1)}{1-\delta} < s_I + f_I^A(1) + f_I^W(1) + \delta s_E + \delta^2 \frac{s_E + f_E^A(1) + f_E^W(1)}{1-\delta},$$

i.e.

$$s_E + \delta f_E^W(1) < s_I + f_I^A(1) + f_I^W(1).$$

Thus, in both cases, we obtain a contradiction. We can therefore focus on the following two pure strategy equilibria:

- All consumers choose I in all periods. This is an equilibrium if and only if no individual consumer has an incentive to deviate. Consider the best possible deviation by an individual consumer. First, that deviation cannot involve any switching from the second period onwards. To see this, suppose the deviating consumer chooses E in the second period. If this deviation ever involves choosing I from the third period onwards, then it must be that $s_E + f_E^W(1) < s_I + f_I^A(1) + f_I^W(1)$. But in that case the consumer would have done better to choose I at least from the second period onwards. Now suppose the deviating consumer chooses I in the second period. If this deviation ever involves choosing E from the third period onwards, then it must be that $s_E + f_E^W(1) \geq s_I + f_I^A(1) + f_I^W(1)$. But in that case the consumer would have done better to choose E at least from the second period onwards. Thus, there are only three possible deviations to consider: all consumers choosing E is an equilibrium if and only if

$$s_I + \delta \frac{s_I + f_I^A(1) + f_I^W(1)}{1-\delta} > \max\left\{ \begin{array}{c} s_E + \delta \frac{s_I + f_I^A(1) + \delta f_I^W(1)}{1-\delta}, \frac{s_E + \delta f_E^W(1)}{1-\delta}, \\ s_I + \delta \frac{s_E + \delta f_E^W(1)}{1-\delta} \end{array} \right\}.$$

This is equivalent to

$$s_E - s_I < \delta \min \left\{ f_I^W(1), f_I^A(1) + f_I^W(1) - f_E^W(1) \right\}.$$

- All consumers buy from E in all periods. By symmetry, this is an equilibrium if and only if

$$s_E - s_I \geq -\delta \min \left\{ f_E^W(1), f_E^A(1) + f_E^W(1) - f_I^W(1) \right\}.$$

Clearly, now there can be multiple equilibria, so beliefs play a role. Furthermore, it is easily verified that

$$-\delta \min \left\{ f_E^W(1), f_E^A(1) + f_E^W(1) - f_I^W(1) \right\} < \delta \min \left\{ f_I^W(1), f_I^A(1) + f_I^W(1) - f_E^W(1) \right\},$$

so an equilibrium always exists.

Bottomline:

- if $s_E - s_I < -\delta \min \left\{ f_E^W(1), f_E^A(1) + f_E^W(1) - f_I^W(1) \right\}$, then the unique equilibrium is for all consumers to choose I in all periods.
- if $s_E - s_I \geq \delta \min \left\{ f_I^W(1), f_I^A(1) + f_I^W(1) - f_E^W(1) \right\}$, then the unique equilibrium is for all consumers to choose E in all periods.
- if $-\delta \min \left\{ f_E^W(1), f_E^A(1) + f_E^W(1) - f_I^W(1) \right\} \leq s_E - s_I < \delta \min \left\{ f_I^W(1), f_I^A(1) + f_I^W(1) - f_E^W(1) \right\}$, then both equilibria exist.

To illustrate that expectations matter, consider Pareto expectations vs. expectations that favor I. Suppose first consumers hold Pareto beliefs, so in case there are multiple equilibria, they coordinate on the equilibrium that yields higher surplus for them. Choosing E for all periods over I yields more consumer surplus if and only if

$$s_E - s_I \geq \delta \left(f_I^A(1) + f_I^W(1) - \left(f_E^A(1) + f_E^W(1) \right) \right).$$

Thus, with Pareto beliefs, E wins if and only if

$$s_E - s_I \geq \min \left\{ \begin{array}{c} \max \left\{ -\delta \min \left\{ f_E^W(1), f_E^A(1) + f_E^W(1) - f_I^W(1) \right\}, \delta \left(f_I^A(1) + f_I^W(1) - \left(f_E^A(1) + f_E^W(1) \right) \right) \right\}, \\ \delta \min \left\{ f_I^W(1), f_I^A(1) + f_I^W(1) - f_E^W(1) \right\} \end{array} \right\}.$$

Meanwhile, if beliefs favor I, then E wins only when it is a unique eqm to choose E, so whenever

$$s_E - s_I \geq \delta \min \left\{ f_I^W(1), f_I^A(1) + f_I^W(1) - f_E^W(1) \right\}.$$

Clearly, the range of parameters over which E wins is larger with Pareto beliefs than with beliefs favoring I, unless

$$f_I^A(1) + f_I^W(1) - \left(f_E^A(1) + f_E^W(1)\right) \geq \min\left\{f_I^W(1), f_I^A(1) + f_I^W(1) - f_E^W(1)\right\},$$

which is equivalent to

$$f_I^A(1) \geq f_E^A(1) + f_E^W(1),$$

i.e. when E's combined learning is less than I's across-user learning. In that case, there is no difference between the outcomes with Pareto beliefs and beliefs favoring I.

Suppose the learning functions are identical, so $f_I^W(1) = f_E^W(1) = f^W(1)$ and $f_I^A(1) = f_E^A(1) = f^A(1)$. Then equilibria are as follows:

- if $s_E - s_I < -\delta \min\left\{f^W(1), f^A(1)\right\}$, then the unique equilibrium is for all consumers to choose I in all periods.
- if $s_E - s_I \geq \delta \min\left\{f^W(1), f^A(1)\right\}$, then the unique equilibrium is for all consumers to choose E in all periods.
- if $-\delta \min\left\{f^W(1), f^A(1)\right\} \leq s_E - s_I < \delta \min\left\{f^W(1), f^A(1)\right\}$, then both equilibria exist.

In this case, with Pareto beliefs, E wins if and only if

$$s_E - s_I \geq 0,$$

while with beliefs favoring I, E wins if and only if

$$s_E - s_I \geq \delta \min\left\{f^W(1), f^A(1)\right\}.$$